

Some Remarks on Units in Grothendieck-Witt Rings

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Abstract

We establish new structures on Grothendieck-Witt rings, including a $GW(k)$ -module structure on the unit group $GW(k)^\times$ and a presentation of \underline{GW}^\times as an infinite \mathbb{G}_m -loop sheaf.

Even though our constructions are motivated by speculations in stable \mathbb{A}^1 -homotopy theory, our arguments are purely algebraic.

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1 Introduction

Summary and Organisation. The main objects of investigation of this article are the ring-valued functors $X \mapsto GW(X)$ and $X \mapsto \underline{GW}(X)$ and their subfunctors of units $GW^\times(X)$ and $\underline{GW}^\times(X)$. Recall that for a scheme X , $GW(X)$ is the *Grothendieck-Witt ring* of X [10], and for X smooth over a perfect field, $\underline{GW}(X)$ is the *unramified Grothendieck-Witt ring* of X [18, Chapter 3]. The connection is that for X (essentially) smooth local, we have $GW(X) = \underline{GW}(X)$ [19, Theorem A].

Here is an overview of the article, which also serves as a summary of our results. At the end of this introduction we give a more leisurely account of some of the main ideas.

In Section 2, we recall the results of Rost and his students on multiplicative transfers for the Grothendieck-Witt ring $GW(X)$ [20, 11]. Specifically, the multiplicative transfer of Rost is defined using a certain norm functor for modules, also defined by Rost. We show that Rost's norm construction coincides with a more general construction of Ferrand [6] (in the situation where both apply).

In Section 3, using this comparison of norm constructions, we show that the assignment $F\acute{e}t/S \ni X \mapsto GW(X)$ defines a Tambara functor. Here $F\acute{e}t/S$ denotes the category of finite étale schemes over S , and by a Tambara functor on this category we mean the evident extension of the notion from [25]; see Definition 4 for details. Using a result of Tambara [25, Theorem 6.1], this also yields an alternative proof that the norm maps extend from $Iso(Bil(\bullet))$ to $GW(\bullet)$.

Section 4 contains our main observation. We show that if k is a field of characteristic not two, then the group of units $GW^\times(k) \subset GW(k)$ is a module over $GW(k)$, in a unique way that is compatible with

the projection formula. By this we mean that if A/k is finite étale, then for $x \in GW(A)$ and $y \in GW^\times(k)$ the following formula holds:

$$y^{tr_{A/k}(x)} = N_{A/k}((y|_A)^x).$$

(We write the module structure as “exponentiation”.) This result is Proposition 15. Uniqueness of the $GW(k)$ -module structure follows from the fact that as an abelian group, $GW(k)$ is generated by the traces of finite étale algebras (in fact traces of degree at most two extensions suffice). Existence/well-definedness is a consequence of Serre’s splitting principle; see Lemma 13. In the remainder of that section we establish many simple but useful properties of this $GW(k)$ -module structure.

In Section 5 we pass to associated sheaves. Thus we study the unramified sheaf of units $\underline{GW}^\times \subset \underline{GW}$. We define a filtration $F_\bullet \underline{GW}^\times$, where $x \in F_n$ if and only if $x \equiv 1 \pmod{\underline{I}^n}$. (Here \underline{I} denotes the unramified sheaf of fundamental ideals.) We can determine the subquotients $F_n \underline{GW}^\times / F_{n+1} \underline{GW}^\times$ and this allows us to prove in Theorem 23 that \underline{GW}^\times is strictly homotopy invariant. This fixes a gap in a result of Wendt [26]. With this preliminary out of the way, we can extend the module structure from the previous section to obtain a \underline{GW} -module structure on \underline{GW}^\times . Using it we define morphisms $\beta_n^\dagger : F_n \underline{GW}^\times \rightarrow (F_{n+1} \underline{GW}^\times)_{-1}$ which we prove in Proposition 26 are isomorphism for $n \geq 2$. In other words, we have constructed a homotopy module F_* with $F_n = F_n \underline{GW}^\times$ for $n \geq 2$. (See Appendix A for some recollections regarding homotopy modules.)

In Section 6 we study the homotopy module F_* . We show that its canonical \underline{GW} -module structure and cohomological transfers coincide respectively (for $* \geq 2$) with the module structure on $F_n \underline{GW}^\times$ constructed in the previous section and Rost’s multiplicative transfers. In doing so we define an isomorphism of homotopy modules $\log : F_* \rightarrow \underline{I}_{tors}^*$. In particular this map turns Rost’s multiplicative transfers into the usual additive ones.

In the final Section 7 we put everything together in Theorem 36. There we construct a homotopy module T_* with $T_0 \cong \underline{GW}^\times$ such that the \underline{GW} -module structure and cohomological transfers on T_0 correspond respectively to the \underline{GW} -module structure and multiplicative transfers on \underline{GW}^\times we have constructed before. Along the way we establish the following exact sequence:

$$0 \rightarrow \underline{I}^2 / 2\underline{I} \xrightarrow{\eta^2, \langle 2 \rangle - 1} \underline{GW} / 2 \oplus \underline{I}_{tors}^2 \rightarrow \underline{GW}^\times \rightarrow 1.$$

This appears to be a novel presentation of the group of units of \underline{GW} .

The paper concludes with two short appendices. In Appendix A we recall the basics about homotopy modules. In Appendix B we recall a well-known continuity result.

Acknowledgements. This work would not have been possible without Rost’s initiation of the study of multiplicative transfers on GW [20] and the detailed computations by his student Wittkop in the case of a quadratic extension [27]. The unpublished preprint of Wendt [26] also was very influential to us (even though we do not end up explicitly re-using any of the results from that paper).

Notations and Conventions. We make the blanket assumption throughout that k is a field of characteristic different from 2. For much of this work we also assume that k is perfect.

The GW -module structure. At the first look, the claim that $GW^\times(k)$ should be a module over $GW(k)$ seems preposterous (at least it did so to the author). Here we try to de-mystify this structure somewhat. First a philosophical remark: one should think as the GW -module structure arising in essentially the same fashion as the \mathbb{Z} -module structure on $\mathbb{Z}^\times = \{\pm 1\}$.

Secondly, here are some formulas. As we have said above the defining property of this GW -module structure is that for $x \in GW^\times(k)$ and A/k finite étale, we have $x^{tr(A)} = N_{A/k}(x|_A)$. Here $tr(A) := tr_{A/k}(1)$. Suppose that $A = k(\sqrt{a})$. Then one may check that $tr(A) = \langle 2 \rangle(1 + \langle a \rangle)$. From this it follows easily that elements of the form $tr(A)$ for $[A : k] \leq 2$ generate $GW(k)$ as an abelian group, so we only have to understand these exponents. Fortunately this situation has been studied thoroughly by Wittkop [27], and the following formula is an immediate corollary of his work (see Lemma 10):

$$(x + y)^{tr(A)} = x^{tr(A)} + y^{tr(A)} + tr(A)xy.$$

Here A/k is an extension of degree two. The trivial extension $A = k \times k$ is allowed, in which case $tr(A) = \langle 2 \rangle(2) = 2$ (see e.g. Lemma 34) and the formula is familiar. From this (and $N_{A/k}(0) = 0$) one also obtains

$$(-1)^{tr(A)} = tr(A) - 1;$$

see Proposition 16 part (ii). Finally one may check the following formula for all $x \in GW(k)$, $a \in k^\times$:

$$\langle a \rangle^x = \langle a \rangle^{\dim(x)}.$$

Together these three formulas in principle allow the computation of x^y for any $x \in GW^\times(k)$ and $y \in GW(k)$. This is illustrated for example in the proof of Proposition 16.

The Logarithm Isomorphism. A further surprising property is that at least on some part of $GW^\times(k)$, the multiplicative structures can be made equivalent to the additive ones. Let us also try to shed some light on that.

The logarithm map furnishes an isomorphism of abelian groups

$$\log : F_2GW^\times(k) = (1 + I_{tors}^2(k), \times) \rightarrow (I_{tors}^2(k), +).$$

This map satisfies

$$\log(xy) = \log(x) + \log(y), \text{ for } x, y \in F_2GW^\times(k),$$

$$\log(x^z) = z \log(x), \text{ for } z \in GW(k)$$

and

$$\log(N_{A/k}(w)) = \text{tr}_{A/k}(\log(w)),$$

for A/k finite étale and $w \in F_2GW^\times(A)$. These three properties are what we mean by turning multiplicative structures into additive ones.

The logarithm map is constructed as follows. Given $x \in F_2GW^\times(k)$, let t_1, \dots, t_m be independent variables. Then consider the element

$$y := x^{(\langle t_1 \rangle - 1) \dots (\langle t_m \rangle - 1)} \in GW^\times(k(t_1, \dots, t_m)).$$

It follows from the theory of homotopy modules that y may be written as $y = 1 + \log_{(m)}(x)(\langle t_1 \rangle - 1) \dots (\langle t_m \rangle - 1)$ for a unique element $\log_{(m)}(x) \in I_{tors}^2(k)$; see Lemma 29. Then $\log(x) := \lim_{m \rightarrow \infty} \log_{(m)}(x)$. This limit makes sense because the sequence is eventually constant; see Theorem 33. See also Remark 30 for a comparison to the logarithm function in real analysis.

2 Multiplicative Transfers on GW

Given a scheme X , we have the categories $Vect(X)$ and $Bil(X)$ of respectively vector bundles on X and bilinear bundles on X provided with a bilinear form. (By “bilinear form” we shall always mean a symmetric, non-degenerate bilinear form.) Write $Iso(Vect(X))$ for the set of isomorphism classes of vector bundles; this is an abelian semi-group. Let $K(Vect(X))^\oplus$ be its associated Grothendieck group. It is also known as the direct-sum K -theory $K_0^\oplus(X)$ of X . If X is affine, this coincides with the usual group $K_0(X)$. We can do the same with $Bil(X)$: we get the abelian semi-group $Iso(Bil(X))$, and the associated Grothendieck group $K(Bil(X))$ coincides with the usual Grothendieck-Witt group $GW(X)$ for X affine.

If $f : X \rightarrow Y$ is a morphism of schemes, there is the usual pushforward $f_* : QCoh(X) \rightarrow QCoh(Y)$. If f is finite locally free and $V \in QCoh(X)$ is a vector bundle, then $f_*(V) \in QCoh(Y)$ is also a vector bundle. Thus there is an induced map $f_* : Iso(Vect(X)) \rightarrow Iso(Vect(Y))$. Since $f_*(E \oplus F) \cong f_*(E) \oplus f_*(F)$, this descends to the Grothendieck group to yield a push-forward homomorphism $tr_f := f_* : K(Vect(X)) \rightarrow K(Vect(Y))$.

If in addition f is étale and $E \in Bil(X)$ then the trace map $f_*\mathcal{O}_X \rightarrow \mathcal{O}_Y$ can be used to turn $f_*E \in Vect(Y)$ into a bilinear bundle. Then as before we obtain $tr_f : K(Bil(X)) \rightarrow K(Bil(Y))$.

The tensor product of vector bundles (or bilinear bundles) turns $Iso(Vect(X))$ (or $Iso(Bil(X))$) into a semi-ring, and $K(Vect(X))$ (or $K(Bil(X))$) into a ring. However, tr_f is not a ring homomorphism: it does not respect multiplication.

This can be remedied to some extent by considering a multiplicative version of transfer.

Given a finite locally free ring homomorphism $R \rightarrow S$, in [6] there is defined a norm functor $N_{S/R} : S\text{-Mod} \rightarrow R\text{-Mod}$. It is lax symmetric monoidal [6, (N6)] and so preserves algebras, modules over algebras, etc.

In order to define this functor, recall that a *polynomial law* between R -modules M_1, M_2 is a natural transformation $\underline{M}_1 \rightarrow \underline{M}_2$, where for an R -module M we put $\underline{M} \in Fun(R\text{-Alg}, Sets)$, $R' \mapsto M \otimes_R R'$

[6, 2.2.1]. In our situation, there is the *norm map* $n_{S/R} : S \rightarrow R, x \mapsto \det(\times x : S \rightarrow S)$. This defines a polynomial law from S to R , because the norm commutes with base change.

If $F \in S\text{-}\mathbf{Mod}$ and $E \in R\text{-}\mathbf{Mod}$, a *norm law* from F to E is a polynomial law ϕ from F (viewed as an R -module) to E such that for every R -algebra R' and $s \in R' \otimes_R S, x \in R' \otimes_R F$ we have $\phi(sx) = n_{R' \otimes_R S/R'}(s)\phi(x)$ [6, Definition 3.2.1].

Given $F \in S\text{-}\mathbf{Mod}$, there is a universal norm law $n_F : F \rightarrow N_{S/R}(F)$ [6, Theorem 3.2.3]. This defines the norm functor.

In geometric language, given a finite locally free morphism $f : X \rightarrow Y$ of affine schemes, we have found a functor $N_f : QCoh(X) \rightarrow QCoh(Y)$. Although we will not need this, let us remark that it is easy to see that the norm construction can be extended to any finite locally free morphism of schemes, affine or not.

The norm functor has nice technical properties, but it can be difficult to get a hold of computationally. There is an alternative construction of Rost [20, 3.2]. For this let $R \rightarrow S$ be locally free of rank n . There exists a canonical homomorphism $\nu_{S/R} : S_n^R S \rightarrow \text{End}_R(\Lambda^n S) \cong R$. Here S_n^R denotes the symmetric powers functor, $S_n^R(M) = (M^{\otimes n})^{S_n}$. In particular $S^{\otimes n}$ has an $S_n^R S$ -module structure, from which we get the homomorphism. We have $\text{End}_R(\Lambda^n S) \cong R$ because $\Lambda^n S$ is an invertible R -module.

Now, given an S -module M we put $\nu_{S/R}(M) = S_n^R M \otimes_{S_n^R S} R$, where the map $S_n^R S \rightarrow R$ is $\nu_{S/R}$ [20, 2.3]. For more details, see [11].

Reverting to geometric language, for a finite étale morphism of constant rank $f : X \rightarrow Y$ between affine schemes, we have found a functor $\nu_f : QCoh(X) \rightarrow QCoh(Y)$. Again, even though we do not need the extra generality, it is easy to see that this construction extends to any finite étale morphism, not necessarily of constant rank, and not necessarily between affine schemes.

Our first task is to show that the above two constructions coincide in sufficiently good cases.

Proposition 1. *Let $f : X \rightarrow Y$ be a finite étale morphism of schemes. Then for $M \in \text{Vect}(X)$ there is a canonical isomorphism $\nu_f(M) \cong N_f(M)$.*

Proof. Canonicity of the isomorphism ensures that it can be glued in open covers. Thus we may assume that $Y = \text{Spec}(R), X = \text{Spec}(S)$ and $R \rightarrow S$ is finite étale of rank n . Then M corresponds to a locally free S -module which we still denote by M .

There is an evident map $\alpha_M : M \rightarrow \nu_{S/R}(M), m \mapsto (m \otimes m \otimes \cdots \otimes m) \otimes 1$. To be clear, this is not a homomorphism. I claim that it defines a norm law. We first need to show that it is even a polynomial law. Hence if $R_1 \rightarrow R_2$ is a homomorphism of R -algebras, we need to show that the following square commutes:

$$\begin{array}{ccc} M \otimes_R R' & \xrightarrow{\alpha_{M \otimes_R R'}} & \nu_{S \otimes_R R'/R'}(M) \\ \downarrow & & \downarrow \\ M \otimes_R R'' & \xrightarrow{\alpha_{M \otimes_R R''}} & \nu_{S \otimes_R R''/R''}(M). \end{array}$$

Here the left vertical map is the induced one. We have $S_n^{R'}(M \otimes_R R') \cong S_n^R(M) \otimes_R R'$ [11, Korollar 2.3.2] and consequently $\nu_{S \otimes_R R'/R'}(M) \cong \nu_{S/R}(M) \otimes_R R'$. The right vertical map is the one induced by this isomorphism.

Checking commutativity is then routine. The proof that this is a norm law boils down to the claim that for $s \in R$ we have $\nu_{S/R}(s \otimes s \otimes \cdots \otimes s) = n_{S/R}(s) \in R$. This follows from [11, Korollar 4.1.2] and [6, (N1)].

By universality of the norm law $M \rightarrow N_{S/R}(M)$ there exists a unique R -linear map $N_{S/R}(M) \rightarrow \nu_{S/R}(M)$ such that the composite $M \rightarrow N_{S/R}(M) \rightarrow \nu_{S/R}(M)$ is α_M . We claim that this is an isomorphism. To see this, we may perform a faithfully flat base change and assume that $S \cong R^d$. But then $M \cong \prod_{i=1}^d M_i$ and $N(M) \cong \bigotimes_i M_i \cong \nu(M)$ [6, Lemme 3.2.4] [11, Satz 4.3.2] (this is where we need M locally free). \square

Let us separate out the last part of the argument:

Proposition 2. *Let $f : X \amalg X \amalg \cdots \amalg X \rightarrow X$ be the fold map of an n -fold coproduct. Then $N_f : \text{Vect}(X \amalg \cdots \amalg X) \cong \text{Vect}(X)^n \rightarrow \text{Vect}(X)$ is given by $(E_1, \dots, E_n) \mapsto E_1 \otimes \cdots \otimes E_n$.*

If $f : X \rightarrow Y$ is finite étale and $E \in \text{Vect}(X)$, then $\nu_f(E) \in \text{Vect}(Y)$, as follows for example from [11, Satz 4.3.2]. The functor $\nu_f = N_f : \text{Vect}(X) \rightarrow \text{Vect}(Y)$ is symmetric monoidal [11, Korollar 4.3.4]. Since $N_f(\mathcal{O}_X) = \mathcal{O}_Y$ it follows easily that the functor preserves bilinear bundles. (See also [11, Korollar 4.2.7].) We have thus found $N_f : \text{Iso}(\text{Vect}(X)) \rightarrow \text{Iso}(\text{Vect}(Y))$ and $N_f : \text{Iso}(\text{Bil}(X)) \rightarrow \text{Iso}(\text{Bil}(Y))$ and these are homomorphisms of *multiplicative* monoids.

3 GW as a Tambara Functor

Recall that if $f : X \rightarrow Y$ is any morphism of schemes, the pullback $f^* : QCoh(X) \rightarrow QCoh(Y)$ induces $f^* : Iso(Vect(X)) \rightarrow Iso(Vect(Y))$ and $f^* : Iso(Bil(X)) \rightarrow Iso(Bil(Y))$, and these are homomorphisms of semi-rings, i.e. respect both the multiplicative and additive structure.

We wish to elaborate somewhat on the compatibilities between restriction (f^*), transfer (tr_f) and norm (N_f). It turns out that Tambara [25] has studied precisely this kind of situation.

We write $F\acute{e}t$ for the category of all schemes, with morphisms the finite étale morphisms. For a scheme S , we let $F\acute{e}t/S$ denote the usual slice category. (Recall that any morphism between schemes which are finite étale over S is automatically finite étale.)

If $f : X \rightarrow Y \in F\acute{e}t/S$ is a morphism, then we get as usual a functor $f^* : F\acute{e}t/Y \rightarrow F\acute{e}t/X$. This functor has a right adjoint f_* which in fact coincides with Weil restriction along f . To see this, it suffices to show that if $T \in F\acute{e}t/Y$ then the Weil restriction $R_{Y/X}(T) \in Sch/X$ is finite étale. This is clear from infinitesimal lifting criteria; see [4, Proposition 7.5.5].

The following definitions are modeled in an evident way on [12, Definition 1.4.1].

Definition 3. Given morphisms $A \xrightarrow{q} X \xrightarrow{f} Y$ in $F\acute{e}t/S$, we can build the following commutative diagram (in $F\acute{e}t/Y$ or $F\acute{e}t/S$) called the *exponential diagram generated by $A \rightarrow X \rightarrow Y$* :

$$\begin{array}{ccccc} X & \xleftarrow{q} & A & \xleftarrow{e} & f^*f_*A \\ f \downarrow & & & & p \downarrow \\ Y & \xrightarrow{\cong} & f_*X & \xleftarrow{f_*q} & f_*A. \end{array}$$

The map $e : f^*f_*A \rightarrow A$ is the counit of adjunction. The map $p : f^*f_*A \rightarrow f_*A$ is not a map of X -schemes, but rather a map of Y -schemes. We should thus more appropriately denote it $f_{\#}f^*f_*A \rightarrow f_*A$, where $f_{\#}$ is the left adjoint of f^* (i.e. $f_{\#}$ is the functor forgetting the X -scheme structure). Then $p : f_{\#}f^*f_*A \rightarrow f_*A$ is also obtained by adjunction.

Definition 4. A *Tambara functor F over S* consists of for each $X \in F\acute{e}t/S$ a semi-ring $F(X)$, together with for each $f : X \rightarrow Y \in F\acute{e}t/S$ three maps $f^* : F(Y) \rightarrow F(X)$, $tr_f : F(X) \rightarrow F(Y)$ and $N_f : F(X) \rightarrow F(Y)$, such that:

1. $F(X \coprod Y) \cong F(X) \times F(Y)$, via the canonical map
2. f^* is a homomorphism, tr_f is a homomorphism of additive monoids, and N_f is a homomorphism of multiplicative monoids
3. f^*, tr_f, N_f are transitive in f (i.e. $(fg)^* = g^*f^*$, and so on)
4. Transfer and norm commute with base change, in the sense that given a cartesian square in $F\acute{e}t/S$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & h \downarrow \\ Z & \xrightarrow{k} & W, \end{array}$$

the following square commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{tr_f} & F(Y) \\ g^* \uparrow & & h^* \uparrow \\ F(Z) & \xrightarrow{tr_k} & F(W), \end{array}$$

and similarly with N in place of tr .

5. Given morphisms $Z \xrightarrow{q} X \xrightarrow{f} Y$, the following diagram induced by the associated exponential diagram commutes:

$$\begin{array}{ccccc} F(X) & \xleftarrow{tr_q} & F(A) & \xrightarrow{e^*} & F(f^*f_*A) \\ N_f \downarrow & & & & N_p \downarrow \\ Y & \xrightarrow{\cong} & f_*X & \xleftarrow{tr_{f_*q}} & f_*A. \end{array}$$

Proposition 5. *The assignments $X \mapsto \text{Iso}(\text{Vect}(X))$ and $X \mapsto \text{Iso}(\text{Bil}(X))$ define Tambara functors on S .*

Proof. We need to verify the axioms. Let us explain the general strategy of the proof. For every axiom, we are given several schemes with morphisms between them, together with certain isomorphism classes of vector (or bilinear) bundles, and we need to show that certain diagrams commute which are obtained by applying functorial constructions to our input data. To do so, we first construct a functorial morphism between the outputs of the two constructions, considered just as vector bundles. We need to show that this morphism is an isomorphism, and possibly that it preserves the bilinear structure. Both of these may be checked after faithfully flat base change, so we may assume that $Y = \text{Spec}(A)$ and $X = \text{Spec}(A^n)$. In this situation proving that the functorial morphism is an isomorphism and respects the structure is usually easy.

Moreover, in order to construct the functorial morphisms, it suffices to construct them when X is affine. Indeed functoriality will imply that the morphisms can be glued along open covers (say).

We only give the proof for the most difficult property, namely (5). We know that Weil restriction coincides with the norm construction [6, Proposition 6.2.2]. We are thus given the following diagram of commutative rings (with all maps finite étale homomorphisms)

$$\begin{array}{ccccc} B & \xrightarrow{q} & C & \xrightarrow{e} & N_f C \otimes_A B \\ f \uparrow & & & & \uparrow p \\ A & \xlongequal{\quad} & A & \xrightarrow{r} & N_f C \end{array}$$

together with a vector (or bilinear) bundle M on C and we need to exhibit a functorial isomorphism $N_f M \cong N_p(M \otimes_C (N_f C \otimes_A B))$. Here functorial means compatible with base change, and isomorphism means isomorphism of (bilinear) A -modules. Following the general strategy, we first define a functorial morphism of A -modules, ignoring the bilinear structure.

We shall construct a norm law $B\text{-}\mathbf{Mod} \ni M \rightarrow N_p(M') \in A\text{-}\mathbf{Mod}$, where $M' := M \otimes_C (N_f C \otimes_A B)$. By universality of the norm law $M \rightarrow N_f M$, this induces a homomorphism $N_f M \rightarrow N_p M'$ as desired.

To do this, consider the composite $M \rightarrow M' \rightarrow N_p(M')$, where $M \rightarrow M'$ is $m \mapsto m \otimes 1$ and $M' \rightarrow N_p(M')$ is the norm map. This defines a polynomial law over A because the entire diagram is functorial under base change in A , which follows from [6, (N2)]. To see that this defines a norm law boils down to verifying that for $b \in B$ we have $n_p(eqb) = rn_f(b)$. Since all our modules are locally free, hence flat, we may again check this after any base change along an injective ring homomorphism, e.g. a faithfully flat one.

We may thus assume that $B = A^d$ and $C = \prod_{i=1}^d C_i$. Then $N_f C = \bigotimes_{i=1}^d C_i$. The canonical map $e : C \rightarrow N_f C \otimes_A B = (N_f C)^d$ is given by

$$(c_1, \dots, c_d) \mapsto (c_1 \otimes 1 \otimes \dots, 1 \otimes c_2 \otimes 1 \otimes \dots, \dots, 1 \otimes 1 \otimes \dots \otimes 1 \otimes c_d).$$

The norm maps are $n_f : B = A^d \rightarrow A, (a_1, \dots, a_d) \mapsto a_1 \dots a_d$ and similarly for n_p . Then $n_p e q = r n_f : B \rightarrow N_f C$ follows immediately.

We have thus defined a canonical morphism $N_f M \rightarrow N_p M'$ which we need to show respects the bilinear structure and is an isomorphism. This is more or less clear. For example, we have $M = \prod_{i=1}^d M_i$, where M_i is a C_i -module. Then $N_f M = \bigotimes_{i=1}^d M_i$, where the tensor product is over A . In contrast, $M' = \prod_{i=1}^d (M_i \otimes_{C_i} N_f C)$, and $N_p M'$ is the tensor product of these terms over $N_f C$. Since $M_1 \otimes_{C_1} N_f C = M_1 \otimes C_2 \otimes \dots \otimes C_d$ and so on, it follows that $N_f M \rightarrow N_p M'$ is an isomorphism. It is similarly straightforward to check that if M is given a bilinear structure, then this isomorphism respects the induced bilinear structures. The remaining details are left to the reader, as is the verification of the easier axioms. \square

Remark 6. We can formally invert the sum operation in $\text{Iso}(\text{Vect}(X))$ (or $\text{Iso}(\text{Bil}(X))$) and then obtain the Grothendieck ring $K(\text{Vect}(X))$ (or $K(\text{Bil}(X))$). It is a priori not at all clear that the norm map $N_f : \text{Iso}(\text{Vect}(X)) \rightarrow \text{Iso}(\text{Vect}(Y))$ induces a map $K(\text{Vect}(X)) \rightarrow K(\text{Vect}(Y))$. The main point of [20] is that this indeed works, and the proof is by showing that the norm maps are *polynomial* (in a sense that is a priori stronger than the definition we have used so far) and then showing that polynomial maps descend to Grothendieck groups.

It is also possible to deduce this fact from our proposition. Indeed, any Tambara functor may be “additively completed” (i.e. one may pass to the Grothendieck ring) [25, Theorem 6.1].

Using either of the above mentioned results, we obtain the following.

Corollary 7. *The assignments $X \mapsto K(\text{Vect}(X))$ and $X \mapsto K(\text{Bil}(X))$ define Tambara functors on S .*

4 The GW -module structure on GW^\times

In this section we let k be a base field of characteristic different from two.

We may consider the Tambara functor $K(\text{Bil}(\bullet)) = GW(\bullet)$ on $F\acute{e}t/k$. If A/k is a finite étale algebra, we write $N_{A/k}$ and $tr_{A/k}$ for the multiplicative and additive transfer, and $\bullet|_A$ for the restriction. We put $tr(A) := tr_{A/k}(1)$.

Recall the dimension homomorphism $dim : GW(A) \rightarrow \mathbb{Z}$. Its kernel is called the fundamental ideal and denoted $I(A)$. Since $GW(A)$ is a ring it has a subset of units $GW^\times(A)$. This is a group where the operation is multiplication in the Grothendieck-Witt ring. We put for $n \geq 0$

$$F_n GW^\times(A) = \{x \in GW^\times(A) \mid x \equiv 1 \pmod{I^n(A)}\}.$$

In other words, $F_n GW^\times(A) = (1 + I^n(A)) \cap GW^\times(A)$. Note that the map $\alpha_n : F_n GW^\times(k) \rightarrow I^n(k)/I^{2n}(k), x \mapsto x - 1$ is a homomorphism (where we use the multiplicative group structure on the left and the additive structure on the right).

Since we shall use it all the time, let us make explicit the following well known fact.

Lemma 8 (Arason [1], Satz 3.3). *Let A/k be a finite étale algebra. Then $tr_{A/k}(I^n(A)) \subset I^n(k)$.*

The following well-known result is very useful for computations.

Lemma 9. *Let l/k be an algebraic field extension of odd degree. (That is, if $l/l_0/k$ is a subextension with l_0/k finite, then $[l_0 : k]$ is odd.) For $0 \leq n \leq m \leq \infty$, the restriction map*

$$I^n(k)/I^m(k) \rightarrow I^n(l)/I^m(l)$$

is injective. Here $I^0(k) := GW(k)$ and $I^\infty(k) := 0$.

Proof. In the proof we shall use transfers along finite extensions which are not separable in general. This is OK because Lemma 8 holds in this generality (and is in fact stated in this generality by Arason). In this text we shall only ever apply Lemma 9 to separable extensions anyway.

By continuity (see Corollary 43 in Appendix B), we may assume that l/k is a finite extension.

Let us first show that $I^n(k)/I^{n+1}(k) \rightarrow I^n(l)/I^{n+1}(l)$ is injective. Since transfer preserves I^n by Lemma 8, we get a well-defined map $tr : I^n(l)/I^{n+1}(l) \rightarrow I^n(k)/I^{n+1}(k)$. It suffices to show that the composite $\alpha : I^n(k)/I^{n+1}(k) \rightarrow I^n(l)/I^{n+1}(l) \rightarrow I^n(k)/I^{n+1}(k)$ is injective. By the projection formula, it is given by multiplication by $tr(A)$ (respectively, in the inseparable case, some other element of dimension $[l : k]$). But $tr(A) \in [l : k] + I$ and consequently α is given by multiplication by $[l : k]$. Since $2I(k) \subset I^2(k)$, i.e. $I^n(k)/I^{n+1}(k)$ is an \mathbb{F}_2 -vector space, multiplication by the odd integer $[l : k]$ is injective.

If $0 \leq n \leq m < \infty$ then $I^n(k)/I^m(k) \rightarrow I^n(l)/I^m(l)$ is a morphism of finitely filtered abelian groups which is injective on the subquotients (by what we have just shown), hence injective. The case $m = \infty$ follows from the fact that $\cap_m I^m(k) = 0$ [22, Corollary 4.5.7], or directly from Springer's theorem [22, Corollary 2.5.4]. \square

Together with Proposition 2, the above lemma implies that when verifying equalities or containments for operations built out of norms, it usually suffices to treat the cases of quadratic field extensions and trivial degree two extensions. The quadratic situation can usually be handled using the results of [27].

By a *degree two extension* A of a field k of characteristic different from two we shall always mean a degree two étale extension; this is either a quadratic field extension of k , or the extension $A = k \times k$. Such an extension has a canonical automorphism denoted by $x \mapsto \bar{x}$. If A/k is a quadratic extension then $x \mapsto \bar{x}$ is the non-trivial Galois automorphism. If $A = k \times k$ then we put $(x, y) := (y, x)$. (Then for any A/k of degree two, the formulas $tr_{A/k}(x) = x + \bar{x}$ and $N_{A/k}(x) = x\bar{x}$ are correct for all $x \in A$, and $k \hookrightarrow A$ consists of precisely the invariants of $x \mapsto \bar{x}$.) We then have the following minimal extension of a result of Wittkop we shall use extensively.

Lemma 10 (Wittkop [27]). *Let $\text{char}(k) \neq 2$, A/k of degree two and $x, y \in GW(A)$. Then*

$$N_{A/k}(x + y) = N_{A/k}(x) + N_{A/k}(y) + tr_{A/k}(x\bar{y}).$$

Proof. If A/k is quadratic, then this is [27, Satz 2.5(ii)]. If $A = k \times k$ then $x = (x_1, x_2), y = (y_1, y_2)$ and by Proposition 2 (and its additive analogue) we have

$$N(x + y) = (x_1 + y_1)(x_2 + y_2) = x_1x_2 + y_1y_2 + (x_1y_2 + x_2y_1) = N(x) + N(y) + \text{tr}(x\bar{y}).$$

□

The following is a very basic result. Its proof illustrates nicely how to use Lemmas 9 and 10. We give many details because we shall re-use the technique several times.

Proposition 11. *Let A/k be a finite étale algebra.*

- (i) *If A/k is of degree p , we have $N_{A/k}(I^n(A)) \subset I^{np}(k)$.*
- (ii) *We have $N_{A/k}(F_nGW^\times(A)) \subset F_nGW^\times(k)$.*
- (iii) *In the situation of (ii), moreover the following diagram commutes:*

$$\begin{array}{ccc} F_nGW^\times(A) & \xrightarrow{\alpha_n} & I^n(A)/I^{2n}(A) \\ N_{A/k} \downarrow & & \text{tr}_{A/k} \downarrow \\ F_nGW^\times(k) & \xrightarrow{\alpha_n} & I^n(k)/I^{2n}(k). \end{array}$$

Proof. Let l/k be a finite Galois extension such that all residue fields of A embed in l , H a 2-Sylow subgroup of $G = \text{Gal}(l/k)$ and $k' = l^H/k$ the associated field extension. Let $A' := A \times_k k'$ denote the base change. By Corollary 7 we have a commutative diagram

$$\begin{array}{ccc} GW(A) & \longrightarrow & GW(A') \\ N_{A/k} \downarrow & & N_{A'/k'} \downarrow \\ GW(k) & \longrightarrow & GW(k'). \end{array}$$

For (i), we wish to show that the composite $I^n(k) \rightarrow GW(A) \xrightarrow{N_{A/k}} GW(k) \rightarrow GW(k)/I^{np}(k)$ is zero. (It is not a homomorphism, but this does not matter.) Since $[k' : k] = |G/H|$ is odd, by Lemma 9 we know that $GW(k)/I^{np}(k) \rightarrow GW(k')/I^{np}(k')$ is injective. Hence consulting the commutative diagram, we find that we may assume (replacing k by k') that G is a 2-group.

I claim that in this case, any subextension $l' \subset l$ can be reached from k by a sequence of quadratic extensions. To see this, we may assume by induction that $k \subset l'$ has no intermediate extensions, i.e. $\text{Gal}(l/l') \subset \text{Gal}(l/k)$ is a proper maximal subgroup. Any such subgroup is of index 2 [21, Theorem 4.6], so the claim is proved.

Now $A = l_1 \times \cdots \times l_r$, where by construction each l_i embeds into l . Thus by the claim it can be reached by a sequence of quadratic extensions. Using transitivity of the norm and induction, we are reduced to the case of A/k of degree two.

If $x, y \in I^n(l)$ with $N_{l/k}(x), N_{l/k}(y) \in I^{2n}(k)$ then $N_{l/k}(x + y) = N_{l/k}(x) + N_{l/k}(y) + \text{tr}_{l/k}(x\bar{y}) \in I^{2n}(k)$, by Lemmas 10 and 8. (The conjugation $y \mapsto \bar{y}$ preserves $I^n(l)$.) Moreover $N_{l/k}(-x) = N_{l/k}(-1)N_{l/k}(x) \in I^{2n}(k)$. It is thus enough to show that $N_{l/k}(x) \in I^{2n}(k)$ for an additive generating set of elements x of $I^n(l)$. If A/k is quadratic this follows from [27, Lemma 1.56 and Satz 2.16 (ii),(iv)]. If $A = k \times k$ then this is obvious.

We have thus proved (i). The proof of (ii) proceeds similarly, using the composite $F_nGW^\times(A) \xrightarrow{N_{A/k}} F_nGW^\times(k) \xrightarrow{x \mapsto x-1} GW(k)/I^{2n}(k)$. In the end we need to show that if A/k is of degree 2 and $1 + x \in F_nGW^\times(A)$, then $N_{A/k}(1 + x) \in F_nGW^\times(k)$. Since the norm is multiplicative it is clear that $N(1 + x)$ is invertible, so it suffices to show that $N_{A/k}(1 + x) \in 1 + I^n(k)$. But $N_{A/k}(1 + x) = 1 + \text{tr}_{A/k}(x) + N_{A/k}(x)$ by Lemma 10 again, and $\text{tr}_{A/k}(x), N_{A/k}(x) \in I^n(k)$ by Lemma 8 and part (i).

To prove part (iii), we may again assume that A/k is of degree two. Let $1 + x \in F_nGW^\times(A)$. Then $N_{A/k}(1 + x) = 1 + N_{A/k}(x) + \text{tr}_{A/k}(x)$ by Lemma 10 once more. Since $N_{A/k}(x) \in I^{2n}(k)$ by (i), this concludes the proof. □

Remark 12. For $n = 0$ the statement (iii) of the proposition is not useful. Instead, it follows from Proposition 2 (and the base change formula) that the following diagram commutes:

$$\begin{array}{ccc} GW^\times(A) & \xrightarrow{\dim} & \{\pm 1\} \\ N_{A/k} \downarrow & & \downarrow [A:k] \\ GW^\times(k) & \xrightarrow{\dim} & \{\pm 1\}. \end{array}$$

The following observation will allow us to turn the norm maps into a GW -module structure.

Lemma 13. *Let $A/k, B/k$ be finite étale algebras and $x \in GW(k)$. If $\text{tr}(A) = \text{tr}(B) \in GW(k)$ then $N_{A/k}(x|_A) = N_{B/k}(x|_B) \in GW(k)$.*

Proof. Since $\text{tr}(A) = \text{tr}(B)$ we must have $[A : k] = [B : k]$. It follows from Proposition 2 that $\dim(N_{A/k}(x|_A)) = \dim(x)^{[A:k]} = \dim(N_{B/k}(x|_B))$. It is thus enough to show that $[N_{A/k}(x|_A)] = [N_{B/k}(x|_B)] \in W(k)$. For this we use Serre's splitting principle. For l/k some field extension and $E \in \text{Et}_n(l)$ an étale algebra of degree $n := [A : k]$, let $\phi(E) = [N_{E/l}(x|_E)] \in W(l)$. This defines an invariant in the sense of [7, Definition 1.1], by the base change formula. It follows from [7, Theorem 29.2] that there exist $x_0, \dots, x_n \in W(k)$ (depending only on x , not on E) such that

$$\phi(E) = x_0 + x_1 \lambda^1(\text{tr}(E)) + \dots + x_n \lambda^n(\text{tr}(E)).$$

The claim follows. □

Remark 14. For an explicit example of the last step of the proof, see [27, Satz 2.10].

If $l = k(\sqrt{a})$ then $\text{tr}(l) = \langle 2 \rangle + \langle 2a \rangle$ [27, Lemma 2.3 (ii)]. It follows easily that any element $y \in GW(k)$ may be written as $\text{tr}(A) - \text{tr}(B)$ for $A/k, B/k$ finite étale algebras. Then for $x \in GW^\times(k)$ we put $x^y := N_{A/k}(x|_A)/N_{B/k}(x|_B)$. This division makes sense because the norm preserves units.

Proposition 15. (i) *The element $x^y \in GW^\times(k)$ is well-defined, independent of the choice of representation $y = \text{tr}(A) - \text{tr}(B)$.*

(ii) *This pairing $GW(k) \times GW^\times(k) \rightarrow GW^\times(k)$ turns $GW^\times(k)$ into a $GW(k)$ -module.*

(iii) *Each of the subgroups $F_n GW^\times(k) \subset GW^\times(k)$ is a $GW(k)$ -submodule.*

(iv) *The $GW(k)$ -module $GW^\times(k)$ satisfies the projection formulas: For A/k finite étale, $y_1 \in GW(A), y_2 \in GW(k), x_1 \in GW^\times(A), x_2 \in GW^\times(k)$ we have*

$$N_{A/k}(x_1^{y_2|_A}) = N_{A/k}(x_1)^{y_2}$$

and

$$N_{A/k}((x_2|_A)^{y_1}) = x_2^{\text{tr}_{A/k}(y_1)}.$$

Proof. (i). If $\text{tr}(A) - \text{tr}(B) = \text{tr}(A') - \text{tr}(B')$ then $\text{tr}(A \times B') = \text{tr}(A) + \text{tr}(B') = \text{tr}(A') + \text{tr}(B) = \text{tr}(A' \times B)$ and consequently

$$N_{A/k}(x|_A)N_{B'/k}(x|_{B'}) = N_{A \times B'/k}(x|_{A \times B'}) = N_{A' \times B/k}(x|_{A' \times B}) = N_{A'/k}(x|_{A'})N_{B/k}(x|_B),$$

where for the middle equality we have used Lemma 13 (and for the outer equalities we use transitivity of the norm as well as the fact that norm along fold maps is multiplication). The claim follows upon division by $N_{B'/k}(x|_{B'})N_{B/k}(x|_B)$.

(ii). We have $(x_1 x_2)^y = N_{A/k}(x_1 x_2)/N_{B/k}(x_1 x_2) = N_{A/k}(x_1)/N_{B/k}(x_1)N_{A/k}(x_2)/N_{B/k}(x_2) = x_1^y x_2^y$, by multiplicativity of the norm. Here $y = \text{tr}(A) - \text{tr}(B)$. In order to prove that $x^{y_1 + y_2} = x^{y_1} x^{y_2}$ it is enough to show that if $A/k, B/k$ are finite étale then $N_{A/k}(x|_A)N_{B/k}(x|_B) = N_{A \times B/k}(x|_{A \times B})$ (since $\text{tr}(A \times B) = \text{tr}(A) + \text{tr}(B)$). This follows from Proposition 2. Since $\text{tr}(k) = 1$ we find $x^1 = N_{k/k}(x) = x$. It remains to show that $x^{yz} = (x^y)^z$. Because of what we have already established, for this it is enough to show that if $A/k, B/k$ are finite étale then $x^{\text{tr}(A)\text{tr}(B)} = (x^{\text{tr}(A)})^{\text{tr}(B)}$. Note that $\text{tr}(A \otimes_k B) = \text{tr}(A)\text{tr}(B)$. Let $t \in GW(A)$. Then by the base change formula (see Definition 4 part (4) and Corollary 7) we get $(N_{A/k}(t))|_B = N_{A \otimes_k B/B}(t|_{A \otimes_k B})$. Substituting $t = x|_A$, applying $N_{B/k}$ and using transitivity of the norm we get $N_{B/k}((N_{A/k}(x|_A))|_B) = N_{A \otimes_k B/k}(x|_{A \otimes_k B})$. This is the desired result.

(iii). It suffices to show that for A/k finite étale we have $N_{A/k}(F_n GW^\times(k)|_A) \subset F_n GW^\times(k)$. This is immediate from Proposition 11.

(iv). We may assume that $y_1 = \text{tr}_{B/A}(1)$, for B/A finite étale, since both sides are linear in y_1 . Then $N_{A/k}((x_2|_A)^{y_1}) = N_{A/k}(N_{B/A}(x_2|_B))$ which equals $N_{B/k}(x_2|_B)$ by transitivity, which is the same as $x_2^{\text{tr}_k(B)}$ by definition. The second claim follows since $\text{tr}_k(B) = \text{tr}_{A/k}(\text{tr}_{B/A}(1))$, by transitivity of transfer. For the first claim, we may assume that $y_2 = \text{tr}(C)$, with C/k finite étale. Then $y_2|_A = \text{tr}_{C \otimes_k A/A}(1)$, by the base change formula (for additive transfers). Thus $N_{A/k}(x_1^{y_2|_A}) = N_{A/k}(N_{C \otimes_k A/A}(x_1|_{C \otimes_k A}))$. By transitivity of the norm, this is the same as $N_{C \otimes_k A/k}(x_1|_{C \otimes_k A}) = N_{C/k}N_{C \otimes_k A/C}(x_1|_{C \otimes_k A})$. By using the base change formula again, we deduce that $N_{C \otimes_k A/C}(x_1|_{C \otimes_k A}) = N_{A/k}(x_1)|_C$. Putting everything together, we find that $N_{A/k}(x_1^{y_2|_A}) = N_{C/k}N_{A/k}(x_1)|_C = (N_{A/k}(x_1))^{y_2}$. This was to be shown. \square

Proposition 16. (i) Suppose that $1 + x \in F_n GW^\times(k)$, and $y \in GW(k)$. Then $(1 + x)^y \equiv 1 + xy \pmod{I^{2n}(k)}$.

(ii) If A/k is of degree two, then $(-1)^{\text{tr}(A)} = \text{tr}(A) - 1$.

(iii) If $y \in I(k)$ then $(-1)^y \equiv 1 + y \pmod{I^2}$.

(iv) For any $n, m \geq 0$ we have $(F_n GW^\times(k))^{I^m(k)} \subset F_{n+m} GW^\times(k)$.

Proof. (i). If $y = \text{tr}(A)$ for A/k finite étale, then this is immediate from Proposition 11 part (iii). In general, let $y = \text{tr}(A) - \text{tr}(B)$ and recall that we write $\alpha_n : F_n GW^\times(k) \rightarrow I^n(k)/I^{2n}(k)$ for the canonical homomorphism. We wish to prove that $\alpha_n((1 + x)^y) = xy$. We have $(1 + x)^y = (1 + x)^{\text{tr}(A)} / (1 + x)^{\text{tr}(B)}$ and hence $\alpha_n((1 + x)^y) = \alpha_n((1 + x)^{\text{tr}(A)}) - \alpha_n((1 + x)^{\text{tr}(B)})$, since α_n is a group homomorphism. By the first sentence, this is the same as $\text{tr}(A)x - \text{tr}(B)x = yx$.

(ii). Using Lemma 10, we compute $0 = N_{A/k}(0) = N_{A/k}(1 + (-1)) = 1 + N_{A/k}(-1) - \text{tr}_{A/k}(1)$. The result follows by rearranging.

(iii). Let $y \in GW(k)$. It follows from Remark 12 that we have $\dim((-1)^y) = (-1)^{\dim(y)}$. Hence if $y \in I(k)$ then $\dim((-1)^y) = 1$ and so $(-1)^y \in F_1 GW^\times(k)$. We now have the two maps $f, g : I(k) \rightarrow I(k)/I(k)^2$ given by $f(y) = [y]$ and $g(y) = \alpha_1((-1)^y)$, and we wish to show that they are equal. Both are group homomorphisms, so we need only check this on generators. Generators of $I(k)$ are given by $\text{tr}(A) - 2$ for A/k quadratic. But for such A by (ii) we have $(-1)^{\text{tr}(A) - 2} = (-1)^{\text{tr}(A)} = \text{tr}(A) - 1 = 1 + (\text{tr}(A) - 2)$. The claim follows.

(iv). The case $m = 0$ is Proposition 15 part (iii). The case $m > 1$ follows from $m = 1$ and induction. So suppose $m = 1$; i.e. we need to show that $(F_n GW(k)^\times)^{I(k)} \subset F_{n+1} GW^\times(k)$.

For $n \geq 1$ the is immediate from (i). Any element of $GW^\times(k)$ can be written as $\pm x$ with $x \in F_1 GW^\times(k)$, and hence the case $n = 0$ follows by combining (iii) and (i). This concludes the proof. \square

Remark 17. It follows from part (iv) of Proposition 16 that $(-1)^{I^2(k)} \subset F_2 GW^\times(k)$. If $\sqrt{2} \in k$ then one may show that actually $(-1)^{I^2(k)} = 1$, but this does not hold in general. See also Theorem 36 and the remark thereafter.

We can use the results of this section to give a kind of presentation of $GW^\times(k)$.

Proposition 18. The $GW(k)$ -module $GW^\times(k)$ is generated by $F_2 GW^\times(k)$ and -1 . Moreover the following sequence is exact:

$$I^2(k) \xrightarrow{p, q} GW(k)/2 \oplus F_2 GW^\times(k) \xrightarrow{r/s} GW^\times(k) \rightarrow 1$$

Here p is the canonical map, $q(y) = (-1)^y$, $r(x) = (-1)^x$ and s is the canonical inclusion.

Proof. Note that $(-1)^2 = 1$ so r makes sense. Moreover if $y \in I^2(k)$ then $(-1)^y \in F_2 GW^\times(k)$ by Proposition 16 part (iv) (or (iii)), so q makes sense.

To show the claim about generation, or equivalently surjectivity of r/s , it suffices to show that any $x \in GW^\times(k)$ can be written as $(-1)^y z$, with $y \in GW(k)$ and $z \in F_2 GW^\times(k)$. Certainly $\dim(x) = \pm 1$, so $x = (-1)^n z_1$ for some $n \in \mathbb{Z}$ and $z_1 \in F_1 GW^\times(k)$. Now $\alpha_1(z_1) \in I(k)/I^2(k)$. Pick $t \in I(k)$ with $[t] = -\alpha_1(z_1)$. Then $(-1)^t z_1 \in F_2 GW^\times(k)$ by Proposition 16 part (iii), and so $x = (-1)^{t+n} ((-1)^t z_1)$ is of the required form.

It remains to verify exactness in the middle. Let $x \in GW(k)/2$. It suffices to show that $(-1)^x \in F_2 GW^\times(k)$ only if x is in the image of p . Hence suppose that $(-1)^x \in F_2 GW^\times(k)$. Then $1 = \dim((-1)^x) = (-1)^{\dim(x)}$ and so $\dim(x)$ is even, whence we may assume that $x \in I$. Now $0 = \alpha_1((-1)^x) = [x]$ by Proposition 16 part (iii) again, and so $x \in I^2$. This concludes the proof. \square

Remark 19. For a more optimal form of this proposition, see Theorem 36.

5 The Sheaf \underline{GW}^\times and the Homotopy Module F_*

Fix a perfect base field k . For most of the section, we will also assume that $\text{char}(k) \neq 2$.

There is the presheaf $\text{Sm}(k) \ni X \mapsto \underline{GW}(X)$. Here $\text{Sm}(k)$ denotes the category of smooth k -schemes. The associated sheaf (in the Nisnevich or Zariski topology) is denoted \underline{GW} , is called the sheaf of unramified Grothendieck-Witt groups, and is strictly homotopy invariant. (Recall that a sheaf F on $\text{Sm}(k)$ is called strictly homotopy invariant if the canonical map $H_{\text{Nis}}^p(X, F) \rightarrow H_{\text{Nis}}^p(X \times \mathbb{A}^1, F)$ is an isomorphism for all $X \in \text{Sm}(k)$ and all $p \geq 0$.) It coincides with the sheaf constructed by Morel [18, Section 3.2] [19, Theorem A]. It is a sheaf of rings. We write \underline{GW}^\times for its subsheaf of units. This is the sheaf associated with $\text{Sm}(k) \ni X \mapsto \underline{GW}(X)^\times$. Our first task is to prove that \underline{GW}^\times is also strictly homotopy invariant.

In order to do this, we recall that there are the sheaves of ideals $\underline{I}^n \subset \underline{GW}$. We define a filtration of \underline{GW}^\times via $F_n \underline{GW}^\times(X) = (1 + \underline{I}^n(X)) \cap \underline{GW}^\times(X)$. As before we get homomorphisms $\alpha_n : F_n \underline{GW}^\times \rightarrow \underline{I}^n / \underline{I}^{2n}$, where on the right hand side we mean the quotient (Nisnevich) sheaf.

If F is any (pre)sheaf on $\text{Sm}(k)$ we write F_{tors} for the (pre)sheaf $F_{\text{tors}}(X) = F(X)_{\text{tors}}$, where for an abelian group A we write A_{tors} for the subgroup of torsion elements. It is strictly homotopy invariant if F is. This follows from the fact that the category of strictly homotopy invariant sheaves is abelian and closed under filtered colimits.

Lemma 20. *If $n \geq 2$ then $F_n \underline{GW}^\times = 1 + \underline{I}_{\text{tors}}^n$.*

Proof. Let $1 + x \in 1 + \underline{I}^n(X) \subset \underline{GW}^\times(X)$. We need to show that $1 + x \in \underline{GW}^\times(X)$ if and only if x is torsion. I claim that x is torsion if and only if it is nilpotent. Indeed since \underline{GW} is strictly homotopy invariant it is unramified [17, Lemma 6.4.4], and thus it suffices to prove the claim for $\underline{GW}(K)$ with K a field, where it follows from [13, Theorems III.3.6 and III.3.8]. We thus need to show that $1 + x \in \underline{GW}^\times(X)$ is invertible if and only if x is nilpotent. Certainly if x is nilpotent then $1 + x$ is invertible. Conversely, if $1 + x$ is invertible then so is its image in $\underline{GW}(K)$ for any field K , and then by unramifiedness of \underline{GW} again it suffices to prove: if $1 + x \in \underline{GW}(K)$ is invertible with $x \in \underline{I}^n(K)$ and $n \geq 2$, then x is nilpotent (or equivalently, torsion).

Let $\sigma : \underline{GW}(K) \rightarrow \mathbb{Z}$ be a signature map. By [13, Theorems III.3.6 and III.3.8] again it suffices to show that $\sigma(x) = 0$. But $\sigma(\underline{I}) \subset 2\mathbb{Z}$ and hence $\sigma(x) \in 2^n\mathbb{Z}$, whereas also $\sigma(1 + x) = 1 + \sigma(x) \in \mathbb{Z}^\times = \{\pm 1\}$. As $n \geq 2$ this implies that $\sigma(x) = 0$, as was to be shown. \square

Lemma 21. *We have $\underline{GW}^\times / F_1 \underline{GW}^\times \cong \mathbb{Z}/2 \cong \mathbb{k}_0^M$, $F_1 \underline{GW}^\times / F_2 \underline{GW}^\times \cong \mathbb{G}_m/2 \cong \mathbb{k}_1^M \cong \underline{I} / \underline{I}^2$ (induced by α_1) and for $n \geq 2$ we have $F_n \underline{GW}^\times / F_{n+1} \underline{GW}^\times \cong \underline{I}_{\text{tors}}^n / \underline{I}_{\text{tors}}^{n+1} \hookrightarrow \underline{I}^n / \underline{I}^{n+1}$ (induced by α_n). In particular all of the subquotients of the filtration are strictly homotopy invariant.*

Proof. For $n = 0$ the map $\underline{GW}^\times / F_1 \underline{GW}^\times \rightarrow (\underline{GW} / \underline{I})^\times = \mathbb{Z}/2$ is an isomorphism: it is surjective since it has a section and it is injective because $F_1 \underline{GW}^\times = (1 + \underline{I}) \cap \underline{GW}^\times$ by definition.

For $n \geq 1$ the map α_n satisfies $\alpha_n^{-1}(\underline{I}^{n+1} / \underline{I}^{2n}) = F_{n+1} \underline{GW}^\times$ and hence induces an injection $\beta_1 : F_n \underline{GW}^\times / F_{n+1} \underline{GW}^\times \hookrightarrow \underline{I}^n / \underline{I}^{n+1}$.

There is a homomorphism $\mathbb{G}_m/2 \rightarrow \underline{GW}^\times, a \mapsto \langle a \rangle$ splitting β_1 , so β_1 is an isomorphism. For $n \geq 2$ by Lemma 20 we have $F_n \underline{GW}^\times = 1 + \underline{I}_{\text{tors}}^n$ and so the image of β_n is $\underline{I}_{\text{tors}}^n / \underline{I}_{\text{tors}}^{n+1} \subset \underline{I}^n / \underline{I}^{n+1}$, as claimed.

For the last claim, since each \underline{I}^n is strictly homotopy invariant [18, Example 3.34] so is $\underline{I}_{\text{tors}}^n$, and hence so is the quotient $\underline{I}_{\text{tors}}^n / \underline{I}_{\text{tors}}^{n+1}$. Here we have used again that the category of strictly homotopy invariant sheaves is abelian and closed under filtered colimits. \square

We will repeatedly use the following result, essentially due to Elman and Lum.

Lemma 22 (Elman and Lum [5]). *Let k be a field of virtual 2-étale cohomological dimension $\text{vcd}_2(k) < n$. Then $\underline{I}_{\text{tors}}^n(k) = 0$ and in particular $2^n \underline{I}_{\text{tors}}^r(k) = 0$ for all $r > 0$ (and also $2^n W_{\text{tors}}(k) = 0$).*

Proof. Applying (vi) of the last theorem of [5] to $K = k(T)$, $F = k$ gives the first statement, at least if $\text{char}(k) \neq 2$. If $\text{char}(k) = 2$ then k is non-orderable, so $\text{cd}_2(k) = \text{vcd}_2(k)$ and $\underline{I}^n(k) = 0$ as a consequence of Voevodsky's resolution of the Milnor conjectures. See [14] for an overview. The remainder follows from $2 \in \underline{I}(k) \subset W(k)$. \square

Theorem 23. *Let k be any field with $\text{char}(k) \neq 2$. Then the sheaf \underline{GW}^\times (on $\text{Sm}(k)$) is strictly homotopy invariant. The same is true for $F_r \underline{GW}^\times$ for any r .*

Proof. Suppose first that $\text{vcd}_2(k) < n$ and k is perfect. Then for any field K/k of transcendence degree at most m over k we have $\text{vcd}_2(k) < n + m$ [23, Theorem 28 of Chapter 4] and so $I_{\text{tors}}^{n+m}(K) = 0$ by Lemma 22.

Let $X \in \text{Sm}(k)$ be of dimension at most m . It follows from the first paragraph and unramifiedness that $F_{n+m}\underline{GW}^\times|_{X_{\text{Nis}}} = 1$. Hence on X (and on $X \times \mathbb{A}^1$) the sheaf \underline{GW}^\times is a *finite* extension of strictly homotopy invariant sheaves, by Lemma 21, and consequently is strictly homotopy invariant.

The same argument works for $F_r\underline{GW}^\times$ for $r \neq 0$.

For the general case in which $\text{vcd}_2(k)$ might be infinite and k need not be perfect we use a continuity argument. Let $k_0 \subset k$ be the prime subfield and write $p : \text{Spec}(k) \rightarrow \text{Spec}(k_0)$ for the canonical morphism. Then k_0 is perfect and $\text{vcd}_2(k) < \infty$. The morphism p is essentially smooth by [9, Lemma A.2]. Hence by Lemma 43 we find that $\underline{GW}|_{\text{Sm}(k)} = p^*(\underline{GW}|_{\text{Sm}(k_0)})$ and thus also $F_r\underline{GW}^\times|_{\text{Sm}(k)} = p^*(F_r\underline{GW}^\times|_{\text{Sm}(k_0)})$. Since p^* preserves strictly homotopy invariant sheaves [9, Lemma A.4], this concludes the proof. \square

Proposition 24. *If $\text{char}(k) \neq 2$ then there exists a unique structure of a \underline{GW} -module on \underline{GW}^\times such that for a field K/k of finite transcendence degree, the induced $GW(K)$ -module structure on $GW^\times(K)$ is the one from Section 4.*

Proof. Uniqueness follows from unramifiedness of \underline{GW} . For existence, let $x \in \underline{GW}(X)$ and $y \in \underline{GW}^\times(X)$. Write $a : X^{(0)} \rightarrow X$ for the inclusion of the generic points. We need to show that $(a^*y)^{a^*x} \in \underline{GW}^\times(X) \subset \underline{GW}^\times(X^{(0)})$. By unramifiedness (in the strong form that $F(X) = \bigcap_{x \in X^{(1)}} F(X_x) \subset F(k(X))$ for an unramified sheaf F and X connected), for this we may assume that X is the spectrum of a dvr (or more generally local ring). In this case $\underline{GW}(X) = GW(X)$ is generated by the one-dimensional diagonal forms $\langle a \rangle$ with $a \in \mathcal{O}^\times(X)$ [13, Corollary I.3.4] and consequently the traces of étale X -schemes generate $GW(X)$ (by the same argument as before). Let $x = \text{tr}(Y_1) - \text{tr}(Y_2)$. Then $(a^*y)^{a^*x} = a^*(N_{Y_1/X}(y|_{Y_1})/N_{Y_2/X}(y|_{Y_2}))$. Since $N_{Y_1/X}(y|_{Y_1}), N_{Y_2/X}(y|_{Y_2}) \in GW(X)^\times$, this concludes the proof. \square

From now we shall assume that $\text{char}(k) \neq 2$.

By unramifiedness, the results from Section 4 over fields immediately extend to all of \underline{GW} :

Corollary 25. (i) *Each of the subsheaves $F_n\underline{GW}^\times \subset \underline{GW}^\times$ is a sub- \underline{GW} -module.*

(ii) *For any $n, m \geq 0$ we have $(F_n\underline{GW}^\times)^{\mathbb{L}^m} \subset F_{n+m}\underline{GW}^\times$.*

Proof. For (i), if $x \in \underline{GW}(X)$ and $y \in F_n\underline{GW}^\times(X)$ then we wish to show that $y^x \in F_n\underline{GW}^\times(X)$. But $F_n\underline{GW}^\times(X) = 1 + \mathbb{L}^n(X) \cap \underline{GW}^\times(X)$ and $\mathbb{L}^n(X) = \mathbb{L}^n(X^{(0)}) \cap \underline{GW}(X)$. (See Lemma 40 for a proof.) Consequently $F_n\underline{GW}^\times(X) = \underline{GW}^\times(X) \cap F_n\underline{GW}^\times(X^{(0)})$. Since $y^x \in \underline{GW}^\times(X)$ by Proposition 24 we are reduced to showing that $y^x \in F_n\underline{GW}^\times(X^{(0)})$. This is Proposition 15 part (iii).

The argument for (ii) is the same, using Proposition 16 part (iv). \square

We can use the \underline{GW} -module structure to define a pairing

$$\beta : \mathbb{ZG}_m \otimes \underline{GW}^\times \rightarrow \underline{GW}^\times, (u, x) \mapsto x^{\langle u \rangle - 1}.$$

Since $\langle u \rangle - 1 \in \mathbb{L}$, by Corollary 25 part (ii) we know that $\beta(\mathbb{ZG}_m \otimes F_n\underline{GW}^\times) \subset F_{n+1}\underline{GW}^\times$. We write $\beta_n : \mathbb{ZG}_m \otimes F_n\underline{GW}^\times \rightarrow F_{n+1}\underline{GW}^\times$ for this restricted pairing.

Recall that for any (pre)sheaf F , its *contraction* is

$$F_{-1} := \underline{\text{Hom}}(\mathbb{ZG}_m, F) \in \text{Pre}(\text{Sm}(k)).$$

By adjunction, β_n induces a homomorphism $\beta_n^\dagger : F_n\underline{GW}^\times \rightarrow (F_{n+1}\underline{GW}^\times)_{-1}$.

Proposition 26. *Let k be a perfect field of characteristic different from 2. For $n \geq 2$ the homomorphism $\beta_n^\dagger : F_n\underline{GW}^\times \rightarrow (F_{n+1}\underline{GW}^\times)_{-1}$ is an isomorphism.*

Proof. The commutative square

$$\begin{array}{ccc} \mathbb{ZG}_m \otimes F_{n+1}\underline{GW}^\times & \xrightarrow{\beta_{n+1}} & F_{n+2}\underline{GW}^\times \\ \downarrow & & \downarrow \\ \mathbb{ZG}_m \otimes F_n\underline{GW}^\times & \xrightarrow{\beta_n} & F_{n+1}\underline{GW}^\times \end{array}$$

induces by adjunction a commutative square

$$\begin{array}{ccc} F_{n+1}\underline{GW}^\times & \xrightarrow{\beta_{n+1}^\dagger} & (F_{n+2}\underline{GW}^\times)_{-1} \\ \downarrow & & \downarrow \\ F_n\underline{GW}^\times & \xrightarrow{\beta_n^\dagger} & (F_{n+1}\underline{GW}^\times)_{-1} \end{array}$$

Since contraction is an exact operation [18, Lemma 7.33], by Lemma 21 we get a diagram of short exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & F_{n+1}\underline{GW}^\times & \longrightarrow & F_n\underline{GW}^\times & \longrightarrow & \underline{I}_{tors}^n / \underline{I}_{tors}^{n+1} \longrightarrow 0 \\ & & \beta_{n+1}^\dagger \downarrow & & \beta_n^\dagger \downarrow & & \gamma_n^\dagger \downarrow \\ 1 & \longrightarrow & (F_{n+2}\underline{GW}^\times)_{-1} & \longrightarrow & (F_{n+1}\underline{GW}^\times)_{-1} & \longrightarrow & (\underline{I}_{tors}^{n+2} / \underline{I}_{tors}^{n+1})_{-1} \longrightarrow 0. \end{array} \quad (1)$$

I claim that γ_n^\dagger is an isomorphism. To see this, let $\delta_n : \mathbb{Z}G_m \otimes \underline{I}^n \rightarrow \underline{I}^{n+1}$ be the homomorphism $(u, x) \mapsto (\langle u \rangle - 1)x$. Then $\delta_n^\dagger : \underline{I}^n \rightarrow (\underline{I}^{n+1})_{-1}$ is an isomorphism. This is just because \underline{I}^* is a homotopy module where the element $[u] \in \underline{K}_1^{MW}$ acts via $\langle u \rangle - 1 \in \underline{I}^1$. It follows that the restriction $\delta_n^\dagger : \underline{I}_{tors}^n \rightarrow (\underline{I}_{tors}^{n+1})_{-1}$ is also an isomorphism (note that $(F_{tors})_{-1} = (F_{-1})_{tors}$). Now consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & \underline{I}_{tors}^{n+1} & \longrightarrow & \underline{I}_{tors}^n & \longrightarrow & \underline{I}_{tors}^n / \underline{I}_{tors}^{n+1} \longrightarrow 0 \\ & & \delta_{n+1}^\dagger \downarrow & & \delta_n^\dagger \downarrow & & \epsilon_n^\dagger \downarrow \\ 1 & \longrightarrow & (\underline{I}_{tors}^{n+2})_{-1} & \longrightarrow & (\underline{I}_{tors}^{n+1})_{-1} & \longrightarrow & (\underline{I}_{tors}^{n+1} / \underline{I}_{tors}^{n+2})_{-1} \longrightarrow 0. \end{array}$$

We find that ϵ_n is an isomorphism. But also $\epsilon_n^\dagger = \gamma_n^\dagger$. For this it suffices to show that $\epsilon_n = \gamma_n$. Since the target is strictly homotopy invariant, hence unramified, it suffices to show that ϵ_n and γ_n induce the same map on sections over fields. This follows from Proposition 16 part (i). (Here we use that $n \geq 2$.) Hence γ_n^\dagger is an isomorphism as claimed.

Now in order to show that β_n^\dagger is an isomorphism, it suffices to show that for every field K (of finite transcendence degree over k) the section $\beta_n^\dagger(K)$ is an isomorphism (since the kernel and cokernel of β_n^\dagger are strictly homotopy invariant and hence unramified). Suppose first that $vcd_2(k) < \infty$. Then also $vcd_2(K) < \infty$ and for n sufficiently large we have $F_n\underline{GW}^\times(K) = 1$, by the Lemmas 20 and 22. In particular for n sufficiently large $\beta_n^\dagger(K)$ is an isomorphism. We may thus prove that $\beta_n^\dagger(K)$ is an isomorphism for all $n \geq 2$ by descending induction on n . The induction step follows by considering the diagram of exact sequences (1) and using that δ_n^\dagger is an isomorphism, as we established above.

For the general case in which $vcd_2(k)$ may be infinite, let $p : \text{Spec}(k) \rightarrow \text{Spec}(k_0)$ be an essentially smooth morphism to a perfect field with $vcd_2(k_0) < \infty$ (e.g. k_0 the ground field). It follows from Lemma 42 that p^* commutes with contractions, and it follows from Corollary 43 that $p^*F_n\underline{GW}^\times = F_n\underline{GW}^\times$. Then $\beta_n^\dagger = p^*\beta_n^\dagger$ is an isomorphism. \square

We have thus managed to deloop the sheaves $F_n\underline{GW}^\times$ for $n \geq 2$. Recall that a *homotopy module* consists of a sequence of sheaves $F_n \in \text{Shv}_{Nis}(\text{Sm}(k))$ together with isomorphisms $F_n \cong (F_{n+1})_{-1}$, such that each F_n is strictly homotopy invariant. See Appendix A for more on homotopy modules.

Corollary 27. *There is a homotopy module F_* (which is essentially unique) such that for $n \geq 2$ we have $F_n = F_n\underline{GW}^\times$, and such that (also for $n \geq 2$) the isomorphisms $F_n \cong (F_{n+1})_{-1}$ are given by β_n^\dagger .*

6 The Logarithm Isomorphism

Throughout this section we fix a perfect base field k of characteristic different from two.

In this section we shall study in more detail the homotopy module F_* . Recall that if G_* is any homotopy module, then each G_n has the structure of a \underline{GW} -module, and also has transfers along finite étale morphisms known as *cohomological transfers*. (See appendix A.) We shall show that for the homotopy module F_* and $n \geq 2$, the \underline{GW} -module structure on $F_n = F_n\underline{GW}^\times$ coincides with that of Section 5, and that the cohomological transfers coincide with Rost's multiplicative transfers.

The first of these claims is actually relatively easy.

Lemma 28. *For $n \geq 2$ the \underline{GW} -module structure on $F_n = F_n \underline{GW}^\times$ coincides with the module structure defined in Section 5.*

Proof. Let K be a field (of finite transcendence degree over k), $\mathcal{O} \subset K$ a geometric dvr with uniformiser π and residue field κ . Let G_* be any homotopy module. There is the boundary map $\partial^\pi : G_{n+1}(K) \rightarrow G_n(\kappa)$ (with kernel $G_{n+1}(\mathcal{O})$), see again the appendix A. The isomorphism $G_n \rightarrow (G_{n+1})_{-1}$ induces a map $\mathbb{Z}G_m \otimes G_n \rightarrow G_{n+1}$ which we denote $(u, x) \mapsto [u]x$. Then one has

$$\partial^\pi([u\pi]x) = \langle s^*(u) \rangle s^*(x), \quad (2)$$

where $s : \text{Spec}(\kappa) \rightarrow \text{Spec}(\mathcal{O})$ is the inclusion of the closed point. (See Lemma 39 in the appendix for a proof.) If in addition \mathcal{O} is Henselian then s has a section and so s^* is surjective.

Now let κ be a field (of finite transcendence degree over k). It suffices to show that the $\underline{GW}(\kappa)$ -module structure on F_n is the canonical one. Choose an essentially smooth local curve over k with residue field κ (e.g. the localisation of \mathbb{A}^1_κ in the origin). Passing to the Henselization, we obtain a Henselian dvr \mathcal{O} with residue field κ and some fraction field K . Pick a uniformiser π . We wish to show that for all $\bar{u} \in \kappa^\times$ and all $\bar{x} \in F_n(\kappa)$ we have $\langle \bar{u} \rangle \bar{x} = \bar{x}^{\langle \bar{u} \rangle}$ (where on the left hand side we mean the module structure coming from F_* being a homotopy module and on the right hand side we mean the canonical module structure constructed earlier). By the first paragraph, for this it suffices to show that $\partial^\pi([u\pi]x) = s^*(x)^{\langle \bar{u} \rangle}$ for all $x \in F_n(\mathcal{O})$ and all $u \in \mathcal{O}^\times$. We compute

$$[u\pi]x = x^{\langle u\pi \rangle - 1} = x^{\langle u \rangle \langle \pi \rangle - \langle u \rangle + \langle u \rangle - 1} = (x^{\langle u \rangle})^{\langle \pi \rangle - 1} x^{\langle u \rangle - 1} = ([\pi]x^{\langle u \rangle})([u]x).$$

But $\partial^\pi([\pi]z) = s^*(z)$ by equation (2), and ∂^π is a homomorphism with kernel $F_{n+1}(\mathcal{O})$. Since $[u]x \in F_{n+1}(\mathcal{O})$ we thus obtain $\partial^\pi([u\pi]x) = \partial^\pi([\pi]x^{\langle u \rangle}) = s^*(x)^{\langle \bar{u} \rangle}$. This concludes the proof. \square

What we have done so far has the following interesting consequence.

Lemma 29. *Let $n \geq 2, m \geq 0$, $X \in \text{Sm}(k)$, and $x \in F_n \underline{GW}^\times(X)$. Then the element*

$$x^{\langle (t_1) - 1 \rangle \langle (t_2) - 1 \rangle \dots \langle (t_m) - 1 \rangle} \in F_{n+m} \underline{GW}^\times(X \times (\mathbb{A}^1 \setminus 0)^m) \subset \underline{GW}(X \times (\mathbb{A}^1 \setminus 0)^m)$$

may be written as

$$1 + \langle (t_1) - 1 \rangle \langle (t_2) - 1 \rangle \dots \langle (t_m) - 1 \rangle y$$

for a unique $y \in \underline{I}^n(X)$.

This induces a bijection (of sets!) $\log_{(m)} : F_n \underline{GW}^\times(X) \rightarrow \underline{I}_{tors}^n(X)$.

Proof. If G_* is any homotopy module, then

$$G_{n+1}(X \times (\mathbb{A}^1 \setminus 0)) = G_{n+1}(X) \oplus [t_1]G_n(X),$$

where $G_n(X) \rightarrow [t_1]G_n(X)$ is an isomorphism. Moreover in this decomposition, the factor $[t_1]G_n(X)$ consists precisely of those $x \in G_{n+1}(X \times (\mathbb{A}^1 \setminus 0))$ such that $i_1^*(x) = 0$, where $i_1 : X \rightarrow X \times (\mathbb{A}^1 \setminus 0)$ is the inclusion corresponding to $1 \in \mathbb{A}^1$. See Lemma 38 in the appendix for a proof. By induction, the map $[t_1] \dots [t_m] : G_n(X) \rightarrow G_n(X \times (\mathbb{A}^1 \setminus 0)^m)$ is injective, and its image consists of precisely those $x \in G_n(X \times (\mathbb{A}^1 \setminus 0)^m)$ such that for each $r \in \{1, \dots, m\}$ we have $j_r^*(x) = 0$. Here $j_r : X \times (\mathbb{A}^1 \setminus 0)^{m-1} \rightarrow X \times (\mathbb{A}^1 \setminus 0)^m$ is the inclusion corresponding to 1 in the r -th factor $\mathbb{A}^1 \setminus 0$.

Applying this to the homotopy module F_* we find that

$$\alpha : F_n(X) \rightarrow F_{n+m}(X \times (\mathbb{A}^1 \setminus 0)^m), x \mapsto [t_1] \dots [t_m]x = x^{\langle (t_1) - 1 \rangle \dots \langle (t_m) - 1 \rangle}$$

is an injection with image consisting of those $y \in F_{n+m}(X \times (\mathbb{A}^1 \setminus 0)^m) \subset \underline{GW}(X \times (\mathbb{A}^1 \setminus 0)^m)$ such that $j_r^*(y) = 1$ for all r . Since j_r^* is a ring homomorphism, we conclude by Lemma 20 that $\alpha - 1$ is a bijection onto the subset of $\underline{I}_{tors}^{n+m}(X \times (\mathbb{A}^1 \setminus 0)^m)$ consisting of those elements such that $j_r^* = 0$ for all r . Applying the remark from the first paragraph to the homotopy module \underline{I}_{tors}^* (for which $[u]x = (\langle u \rangle - 1)x$) concludes the proof. \square

Thus if $x \in F_n \underline{GW}^\times(X)$ we obtain a sequence $x = \log_{(0)}(x), \log_{(1)}(x), \log_{(2)}(x), \dots \in \underline{I}_{tors}^n(X)$. We would like to take the “limit” of this sequence.

For the remainder of this section, we will use the abbreviation $P_m := \prod_{i=1}^m \langle (t_i) - 1 \rangle$.

Remark 30. Thus $x^{P_m} = 1 + P_m \log_{(m)}(x)$. Since multiplication by P_m is injective in some sense (see the previous proof), we may write this as

$$\log_{(m)}(x) = (x^{P_m} - 1)/P_m.$$

We think of $P_m \in I^m$ as small, and taking the limit we propose corresponds to the formula

$$\lim_{\epsilon \rightarrow 0+} (x^\epsilon - 1)/\epsilon = \log(x)$$

from classical analysis.

Lemma 31. *Let L/K be a degree two extension (K of characteristic not two).*

(i) *We have $N_{L(t)/K(t)}(\langle t \rangle - 1) = -tr(L)(\langle t \rangle - 1)$.*

(ii) *We have $tr(L)^2 = 2tr(L)$.*

(iii) *There exists $y \in GW(K)$ such that $yN_{L/K}(2) = 8$.*

Proof. (i). By Lemma 10 we get $N(\langle t \rangle - 1) = N(\langle t \rangle) - tr(1)\langle t \rangle + N(-1)$. Since $N(\langle t \rangle) = \langle t^2 \rangle = 1$ (this is clear if $A = k \times k$, and for the quadratic case see [27, Lemma 2.6(ii)]) and $N(-1) = tr(1) - 1$ by Proposition 16 part (ii), the result follows from the observation that $tr_{L(t)/K(t)}(1) = tr_{L/K}(1)|_{GW(K(t))}$ (i.e. the base change formula.)

(ii). Since $tr(L) - 1 = N(-1)$ we get $(tr(L) - 1)^2 = 1$. The result follows.

(iii). We have $N(2) = N(1 + 1) = 2 + tr(1)$ by Lemma 10 again. Thus if we put $\xi = N(2)$ then by (ii) we find $(\xi - 2)^2 = 2(\xi - 2)$ which implies that $\xi(6 - \xi) = 8$. \square

Corollary 32. *Let L/K be as above, $x \in GW(L)$ and $2^r x = 0$. Then for $m > 3r$ we have $N_{L(t_1, \dots, t_m)/K(t_1, \dots, t_m)}(1 + P_m x) = 1 + P_m tr_{L/K}(x)$.*

Proof. Using Lemma 10 we compute that $N(1 + P_m x) = 1 + P_m tr(x) + N(P_m)N(x)$. We thus need to show that $N(P_m)N(x) = 0$. Since $2^r x = 0$ we get $0 = N(2^r x) = N(2^r)N(x)$, and hence by Lemma 31 part (iii) we find that $8^r N(x) = 0$. I claim that $N(P_m) = (-1)^m 2^{m-1} tr(L) P_m$. Since $m > 3r$ the claim implies that $N(P_m)$ is divisible by 8^r and hence $N(P_m)N(x) = 0$ as needed.

To prove the claim, note that $N(P_m) = \prod_{i=1}^m N(\langle t_i \rangle - 1) = (-tr(L))^m P_m$, by Lemma 31 part (i), and $(-tr(L))^m = (-1)^m 2^{m-1} tr(L)$ by Lemma 31 part (ii). \square

Theorem 33. *For $n \geq 2$ and $x \in F_n \underline{GW}^\times(X)$, the sequence $\log_{(m)}(x) \in \underline{I}_{tors}^n(x)$ is eventually constant. Write $\log(x)$ for this eventual value.*

This defines an isomorphism of homotopy modules $\log : F_ \rightarrow \underline{I}_{tors}^*$ which intertwines Rost's multiplicative transfer on the left with the canonical transfer on the right. In particular the cohomological transfer on F_* coincides with Rost's multiplicative transfer.*

Proof. It follows from (for example) the base change formulas that each $\log_{(m)}$ is a morphism of sheaves of sets. Hence everything in sight is a morphism of sheaves, and we need not worry about establishing this property.

Suppose that there exists $r > 0$ such that $2^r \underline{I}_{tors}^n(X) = 0$. If $vcd_2(k) < \infty$ such an r exists by unramifiedness of \underline{I}_{tors}^n and Lemma 22.

For $x', y' \in \underline{I}_{tors}^n(X)$ we have $(1 + P_m x')(1 + P_m y') = 1 + P_m(x' + y') + P_m^2 x' y'$. Since $(\langle t_i \rangle - 1)^2 = -2(\langle t_i \rangle - 1)$ we find that P_m^2 is divisible by 2^m , and hence $P_m^2 x' = 0$ if $x' \in \underline{I}_{tors}^n(X)$ and $m > r$. This shows that $\log_{(m)}$ is a morphism of sheaves of abelian groups, for $m > r$.

I claim that for $m > 3r$, $x \in F_n \underline{GW}^\times(X)$ and $y \in \underline{GW}(X)$ we have $y \log_{(m)}(x) = \log_{(m)}(x^y)$, i.e. that $\log_{(m)}$ intertwines the GW -module structures. For this we may write $x^{P_m} = 1 + P_m \log_{(m)}(x)$. Now $(x^y)^{P_m} = (x^{P_m})^y = (1 + P_m \log_{(m)}(x))^y$ and so it is enough to show that $(1 + P_m x')^y = 1 + y P_m x'$ for every $x' \in \underline{I}_{tors}^n(X)$. By unramifiedness, we may assume that X is the spectrum of a field K . Since $GW(K)$ is generated as an abelian group by 1 and the traces of quadratic extensions, and $\log_{(m)}$ is a homomorphism of abelian groups by what we have already done, it suffices to prove that for L/K quadratic we have $N_{L(t_1, \dots, t_n)/K(t_1, \dots, t_n)}(1 + P_m x') = 1 + tr(L) P_m x'$. But $2^r x' = 0$ by assumption, so this follows from Corollary 32.

Now let $m > 3r$. We compute

$$x^{P_{m+1}} = (x^{P_m})^{\langle t_{m+1} \rangle - 1} = (1 + P_m \log_{(m)}(x))^{\langle t_{m+1} \rangle - 1} = 1 + (\langle t_{m+1} \rangle - 1) P_m \log_{(m)}(x),$$

where in the last step we have used the claim.

In other words we have found that $1 + P_{m+1} \log_{(m+1)}(x) = 1 + P_{m+1} \log_{(m)} x$, i.e. the sequence is constant (recall the uniqueness part of Lemma 29). Thus the “limit function” \log exists.

It is clear that \log must be an isomorphism, because each of the maps $\log_{(m)}$ is a bijection.

In order to show that \log intertwines transfers along all étale morphisms $f : S \rightarrow T$ we may by unramifiedness assume that T is the spectrum of a field. Then by Lemma 9 and the argument from Proposition 11 we reduce to transfers along degree two extensions. In this case the result follows from Corollary 32.

We need to show that \log is a morphism of homotopy modules, i.e. is compatible with the isomorphisms $\beta_n^\dagger : F_n \rightarrow (F_{n+1})_{-1}$. To see this it is enough to show that $\log([u]x) = [u] \log(x)$. But

$$\log([u]x) = \log(x^{\langle u \rangle - 1}) = (\langle u \rangle - 1) \log(x) = [u] \log(x).$$

Here the first and last equality are by definition, and the middle one is because \log is a homomorphism of GW -modules, as we have shown above.

We are thus finished with the proof, in the case that $vcd_2(k) < \infty$. For the general case we use the standard continuity argument. It is enough to show that for every field K/k and every $x \in F_n GW^\times(K)$ the sequence $\log_{(m)}(x)$ is eventually constant. Indeed this shows that $\log : F_* \rightarrow \varinjlim_{tors}^*$ is a well-defined map (of sheaves of sets, say). Then by Corollary 43 from Appendix B we have $\log(K) = \text{colim}_l \log(l)$, where the colimit is over $l \subset K$ which are finitely generated over the prime field. Thus $vcd_2(l) < \infty$ for all such l , and hence $\log(l)$ is an isomorphism with all the desired properties (respects module structures and transfers), by what we have done above. Hence so is the colimit $\log(K)$.

Finally if $x \in F_n GW^\times(K)$ then by Corollary 43 again there exist a subfield $l \subset K$ which is finitely generated over the prime field and $y \in F_n GW^\times(l)$, such that $x = y|_K$. By what we have already proved, $\log_{(m)}(y)$ is eventually constant, and hence so is $\log_{(m)}(x) = \log_{(m)}(y|_K) = \log_{(m)}(y)|_K$.

This concludes the proof. \square

7 Delooping \underline{GW}^\times

In this section, we finally put everything together.

Lemma 34. *For any field K of characteristic different from two we have the equality $2(\langle 2 \rangle - 1) = 0 \in GW(K)$.*

Proof. Since $(x+y)^2 + (x-y)^2 = 2(x^2 + y^2)$ we have $\langle 2, 2 \rangle = \langle 1, 1 \rangle$. The result follows. \square

Lemma 35. *The maps $\underline{I}^n \rightarrow F_n \underline{GW}^\times, x \mapsto (-1)^x$ assemble to a morphism of homotopy modules $\underline{I}^* \rightarrow F_*$. The composite*

$$\underline{I}^* \rightarrow F_* \xrightarrow{\log} \varinjlim_{tors}^*$$

is multiplication by $\langle 2 \rangle - 1$.

Proof. We have $(-1)^{\underline{I}^n} \subset F_n \underline{GW}^\times$ by Proposition 16 part (iv), so the statement makes sense. In order to see that we have a morphism of homotopy modules, we need to show that for $x \in \underline{I}^n(K)$ and $u \in K^\times$ we have $(-1)^{[u]x} = [u]((-1)^x)$. Now $[u]x = (\langle u \rangle - 1)x$ and so $(-1)^{[u]x} = ((-1)^x)^{\langle u \rangle - 1} = [u]((-1)^x)$ as needed.

Since $\underline{I}^* = \underline{K}_*^W = \underline{K}_*^{MW}/h$ [16] we know that there must exist $x \in \underline{GW}(k)$ such that the composite is given by multiplication by x . It is enough to prove that the composite $\underline{I}^2 \rightarrow F_2 \underline{GW}^\times \rightarrow \varinjlim_{tors}^2$ is given by multiplication by $\langle 2 \rangle - 1$. Indeed then the two-fold contraction $\underline{W} \rightarrow \underline{W}_{tors}$ will also be given by multiplication by $\langle 2 \rangle - 1$, and $1 \in \underline{W}$ generates \underline{I}^* .

Fix some field K . The ideal $I^2(K)$ is generated by elements of the form $(\langle a \rangle - 1)(\langle b \rangle - 1)$. It is thus enough to prove that

$$(-1)^{(\langle a \rangle - 1)(\langle b \rangle - 1)(\langle t_1 \rangle - 1) \dots (\langle t_m \rangle - 1)} = 1 + (\langle 2 \rangle - 1)(\langle a \rangle - 1)(\langle b \rangle - 1)(\langle t_1 \rangle - 1) \dots (\langle t_m \rangle - 1) \quad (3)$$

for infinitely many m .

Let $A = k(\sqrt{a}), B = k(\sqrt{b})$. Then $\text{tr}(A) = \langle 2 \rangle(\langle a \rangle + 1)$ and similarly for B , and we compute

$$\begin{aligned}
(-1)^{\langle a \rangle - 1 \langle b \rangle - 1} &= (-1)^{\langle a \rangle + 1 \langle b \rangle + 1} \\
&= (-1)^{\text{tr}(A)\text{tr}(B)} \\
&= (\text{tr}(A) - 1)^{\text{tr}(B)} \\
&= (\langle 2 \rangle(\langle a \rangle + 1) - 1)^{\text{tr}(B)} \\
&= (\langle a \rangle + 1)^{\text{tr}(B)} - \text{tr}(A)\text{tr}(B) + (-1)^{\text{tr}(B)} \\
&= 2 + \langle a \rangle \text{tr}(B) - \text{tr}(A)\text{tr}(B) + \text{tr}(B) - 1 \\
&= 1 + \text{tr}(B)(1 + \langle a \rangle)(1 - \langle 2 \rangle) \\
&= 1 + (\langle 2 \rangle - 1)(\langle a \rangle - 1)(\langle b \rangle - 1) \\
&=: 1 + \xi.
\end{aligned}$$

Here we have used that $(-1)^2 = 1 = \langle 2 \rangle^2$, Proposition 16 part (ii), and Lemmas 34 and 10.

I now claim that for any field extension K'/K , and any $y \in GW(K')$ we have $(1 + \xi)^y = 1 + y\xi$. Since $GW(K')$ is generated as an abelian group by 1 and the traces of quadratic extensions, it suffices to show that $\xi^2 = 0$ and $(1 + \xi)^{\text{tr}(T)} = 1 + \text{tr}(T)\xi$ for T/K' quadratic, or equivalently that $\xi^{\text{tr}(T)} = 0$. We have $(\langle a \rangle - 1)^2 = -2(\langle a \rangle - 1)$, and hence $\xi^2 = 0$ follows from $2(\langle 2 \rangle - 1) = 0$, i.e. Lemma 34.

If T/K' is quadratic then $(\langle a \rangle - 1)^{\text{tr}(T)} = 1 - \text{tr}(T)\langle a \rangle + \text{tr}(T) - 1 = -\text{tr}(T)(\langle a \rangle - 1)$. Hence $\xi^{\text{tr}(T)} = (\langle 2 \rangle - 1)^{\text{tr}(T)}(\langle a \rangle - 1)^{\text{tr}(T)}(\langle b \rangle - 1)^{\text{tr}(T)} = -\text{tr}(T)^3\xi = -4\text{tr}(T)\xi = 0$ by Lemmas 34 and 31 again.

Thus the claim is proved. It implies that the equation (3) actually holds for all m . This concludes the proof. \square

Theorem 36. *Let k be a perfect field of characteristic different from 2. There exists a short exact sequence of homotopy modules*

$$0 \rightarrow \underline{K}_{*+2}^W/[-1]\underline{K}_{*+1}^W \xrightarrow{\eta^2, \langle \langle 2 \rangle - 1 \rangle} \underline{K}_*^{MW}/2 \oplus \underline{K}_{\text{tors}, *+2}^W \rightarrow T_* \rightarrow 0,$$

defining T_* . There is a canonical isomorphism $T_0 \cong \underline{GW}^\times$. Via this isomorphism, the \underline{GW} -module structure as well as the cohomological transfers on T_0 coincide with the module structure from Section 5 and Rost's multiplicative transfers, respectively.

Here by $[-1] \in K_{*+2}^W(k) \cong I(k)$ we denote the element $\langle -1 \rangle - 1$, and η^2 is just the natural inclusion $\underline{K}_{*+2}^W = \underline{I}^{*+2} \hookrightarrow \underline{K}_*^{MW} = \underline{I}^* \times_{\underline{K}_*^M} \underline{K}_*^M$. One may note that $\langle -1 \rangle - 1 \in I(k) \subset W(k)$ is the same as $-2 \in W(k)$.

Proof. Note that $(\langle 2 \rangle - 1)[-1] = (\langle 2 \rangle - 1)(\langle -1 \rangle - 1) = (\langle 2 \rangle - 1)(-2) = 0 \in I(K) \subset W(K)$ by Lemma 34, and $\eta^2([-1]) = \eta(-2) = -2\eta$, so the sequence makes sense.

Let us first show that $\underline{K}_{*+2}^W/[-1] \xrightarrow{\eta^2, \langle \langle 2 \rangle - 1 \rangle} \underline{K}_*^{MW}/2 \oplus \underline{K}_{\text{tors}, *+2}^W$ is injective. For this let $x \in K_{n+2}^W(K)$ (we may assume that $n \geq 0$) and suppose that $\eta^2(x) = 0$. Equivalently, we have $x \in I^{n+2}(K)$ with $\eta^2(x) \in 2K_n^{MW}(K)$. Then in particular $x \in 2I^n(K) \subset 2W(K)$. By [2, Theorem 2.2] we conclude that $x \in 2I^{n+1}(K)$ (note that $-2 = \langle -1, -1 \rangle \in W(K)$ is a Pfister form). Consequently $x \equiv 0 \in K_{n+2}^W(K)/[-1]K_{n+1}^W(K)$, i.e. η^2 alone is already injective.

Let us now show that $T_0 \cong \underline{GW}^\times$. For this we consider the map $r/s : \underline{K}_0^{MW}/2 \oplus \underline{K}_{\text{tors}, 2}^W \cong \underline{GW}/2 \oplus F_2\underline{GW}^\times \rightarrow \underline{GW}^\times$. Here $F_2\underline{GW}^\times \cong \underline{K}_{\text{tors}, 2}^W$ is the logarithm isomorphism, $r : \underline{GW}/2 \rightarrow \underline{GW}^\times$ is $x \mapsto (-1)^x$ and $s : F_2\underline{GW}^\times \rightarrow \underline{GW}^\times$ is the inclusion. It follows from Lemma 35 that the composite $\underline{I}^2 \xrightarrow{\langle 2 \rangle - 1} \underline{I}_{\text{tors}}^2 \cong F_2\underline{GW}^\times \hookrightarrow \underline{GW}^\times$ is $x \mapsto (-1)^x$ and hence it follows from Proposition 18 that r/s factors through T_0 and induces an isomorphism (both statements can be checked on sections over fields, since we are working with strictly homotopy invariant sheaves).

Now in order to see that r/s preserves the \underline{GW} -module structure and the transfers, it suffices to consider r and s separately. The fact that this works for r follows from Proposition 15 parts (ii) and (iv). The map s is the composite of the logarithm isomorphism, which preserves the module structure and transfers by Theorem 33, and the morphism of multiplication by a constant (namely $\langle 2 \rangle - 1$) which also preserves the module structure and transfers. \square

Remark 37. If $\sqrt{2} \in k$ (i.e. $\langle 2 \rangle = 1 \in W(k)$) then we get a splitting $T_* \cong \underline{K}_{\text{tors}, *+2}^W \oplus \underline{K}_*^{MW}/(2, \eta^2)$, but not in general. This is essentially the same as Remark 17.

A Recollections on Homotopy Modules

Throughout, k is a perfect base field. We recall some well known facts about homotopy modules which seem hard to find explicitly in the literature. We make no claim to originality.

The Basics. A homotopy module [15, Section 5.2] consists of a collection of strictly homotopy invariant sheaves $F_* \in Shv_{Nis}(Sm(k)), * \in \mathbb{Z}$ together with isomorphisms $F_n \rightarrow (F_{n+1})_{-1}$. A morphism of homotopy modules $\alpha_* : F_* \rightarrow G_*$ consists of morphisms of sheaves $\alpha_n : F_n \rightarrow G_n$ for all n such that $(\alpha_{n+1})_{-1} = \alpha_n$ under the canonical identifications.

The category of homotopy modules is equivalent to the heart of the homotopy t -structure on $\mathbf{SH}(k)$ [15, Theorem 5.2.6]. This implies that they have a lot more structure than is immediately apparent. In this appendix we clarify some of this structure.

\underline{K}_*^{MW} -module Structure. The object $\underline{\pi}_0(\mathbb{1})_* \cong \underline{K}_*^{MW}$ is the unit of a symmetric monoidal structure on the category of homotopy modules. For the definition of the sheaf of unramified Milnor-Witt K -theory \underline{K}_*^{MW} , see [18, Chapter 3]. Its sections over a field L are generated by the classes $[u] \in K_1^{MW}(L)$ for $u \in L^\times$ and $\eta \in K_{-1}^{MW}(L)$. One puts $\langle u \rangle = 1 + \eta[u]$; this induces an isomorphism $K_0^{MW}(L) \cong GW(L)$.

The isomorphism $F_n \rightarrow (F_{n+1})_{-1}$ corresponds by adjunction to a morphism $\mathbb{Z}\mathbb{G}_m \otimes F_n \rightarrow F_{n+1}$. It factors through the surjection $\mathbb{Z}\mathbb{G}_m \rightarrow \underline{K}_1^{MW}$. By contraction, the pairing $\underline{K}_1^{MW} \otimes F_n \rightarrow F_{n+1}$ induces $(\underline{K}_1^{MW})_{-2} \otimes F_n \rightarrow (F_{n+1})_{-2}$ and hence a multiplication $\eta : F_n \rightarrow F_{n-1}$. Since \underline{K}_*^{MW} is generated by \underline{K}_1^{MW} and $\eta \in \underline{K}_{-1}^{MW}$, it follows that there is at most one extension to a pairing $\underline{K}_*^{MW} \otimes F_* \rightarrow F_{*+*}$. It is a consequence of the identification of the category of homotopy modules with the heart of $\mathbf{SH}(k)$ that this extension always exists.

Cohomological Transfers. A homotopy module automatically has cohomological transfers, i.e. for any finite étale morphism $f : X \rightarrow Y$ with X, Y essentially smooth over k , there is a transfer $tr_{X/Y} : F_*(X) \rightarrow F_*(Y)$. See [18, Corollary 5.30] or [3, Section 4]. Cohomological transfers are functorial in morphisms of homotopy modules and satisfy the projection and base change formulas (*loc. cit.*).

Contractions.

Lemma 38. Let t be the coordinate on \mathbb{A}^1 and F_* a homotopy module. Write $i_1 : Spec(k) \rightarrow \mathbb{A}^1 \setminus 0$ for the inclusion of the point 1. Then for $X \in Sm(k)$ we have $F_*(X \times (\mathbb{A}^1 \setminus 0)) \cong F_*(X) \oplus F_{*-1}(X)$. Here the map $F_{*-1}(X) \rightarrow F_*(X \times (\mathbb{A}^1 \setminus 0))$ is multiplication by $[t] \in K_1^{MW}(k[t, t^{-1}])$, the map $F_*(X) \rightarrow F_*(X \times (\mathbb{A}^1 \setminus 0))$ is pullback along the canonical projection, and the subgroup $F_{*-1}(X) \subset F_*(X \times (\mathbb{A}^1 \setminus 0))$ is precisely the kernel of i_1^* .

Proof. We have the inclusions $Spec(k) \xrightarrow{i_1} \mathbb{A}^1 \setminus 0 \rightarrow \mathbb{A}^1$. Since F is homotopy invariant, it follows that $F_*(X \times (\mathbb{A}^1 \setminus 0)) = F_*(X) \oplus M_*(X)$, where $M_*(X)$ is the kernel of i_1^* and by definition of contraction, $M_*(X) = (F_*)_{-1}(X)$. We thus use the defining isomorphism of a homotopy module to identify $M_*(X) \cong F_{*-1}(X)$. It remains to see that this isomorphism is given by multiplication by $[t]$.

For this, let F be any homotopy invariant sheaf. The pairing $\mathbb{G}_m(U) \otimes F_{-1}(U) \rightarrow F(U)$ is given by $(u, s) \mapsto u^*s$. Here $s \in F_{-1}(U) \subset F(U \times (\mathbb{A}^1 \setminus 0))$ and $u \in \text{Hom}_k(U, \mathbb{A}^1 \setminus 0) = \text{Hom}_U(U, U \times (\mathbb{A}^1 \setminus 0))$. Consequently “multiplication by $[t]$ ”

$$F(X \times (\mathbb{A}^1 \setminus 0)) \supset F_{-1}(X) \rightarrow F_{-1}(X \times (\mathbb{A}^1 \setminus 0)) \xrightarrow{[t]} F(X \times (\mathbb{A}^1 \setminus 0))$$

is given by $F(X \times (\mathbb{A}^1 \setminus 0)) \supset F_{-1}(X) \ni s \mapsto t^*p^*(s)$, where $p : X \times (\mathbb{A}^1 \setminus 0)^2 \rightarrow X \times (\mathbb{A}^1 \setminus 0)$ is projection to the first two factors and $t : X \times (\mathbb{A}^1 \setminus 0) \rightarrow X \times (\mathbb{A}^1 \setminus 0)^2$ is $(x, u) \mapsto (x, u, u)$. Thus $pt = \text{id}$ and so multiplication by $[t]$ corresponds to the canonical inclusion, as was to be shown. \square

Boundary Maps. Being strictly homotopy invariant, each of the sheaves F_n is unramified [17, Lemma 6.4.4]. This means that for a dense open immersion $U \rightarrow X \in Sm(k)$, the restriction $F_*(X) \rightarrow F_*(U)$ is injective and that moreover for X connected we have

$$F_*(X) = \bigcap_{x \in X^{(1)}} F_*(X_x).$$

Here $X^{(1)}$ denotes the set of points of codimension one, $F_*(X_x)$ denotes the stalk at x , and the intersection takes place in $F_*(k(X))$.

If L/k is a field extension, then by a standard colimit procedure there is a well-defined group of sections $F_*(L)$. More generally this is true if L is a scheme which is a filtering inverse limit of a system of smooth schemes with affine transition morphisms. If L is a field extension of finite transcendence degree over k and $\mathcal{O} \subset L$ is a geometric dvr, then by definition there exist $X \in \text{Sm}(k)$ and $x \in X^{(1)}$ such that $L \cong k(X)$ and $\text{Spec}(\mathcal{O}) \cong X_x$. Let κ be the residue field of \mathcal{O} . Then for every choice of uniformizer π of \mathcal{O} there exists a canonical *boundary map*

$$\partial^\pi : F_*(L) \rightarrow F_{*-1}(\kappa)$$

with kernel $F_*(\mathcal{O})$ [18, discussion after Corollary 2.35].

In this situation, we write $s : \text{Spec}(\kappa) \rightarrow \text{Spec}(\mathcal{O})$ for the inclusion of the closed point.

Lemma 39. *For $u \in \mathcal{O}^\times$ and $m \in F_*(\mathcal{O})$ we have $\partial^\pi([u\pi]m) = \langle s^*(u) \rangle s^*(m)$.*

Proof. First note that for $x \in K_*^{MW}(K)$ and $m \in F_*(\mathcal{O})$ we have $\partial^\pi(xm) = \partial^\pi(x)s^*(m)$. To see this, one goes back to the geometric construction of ∂^π as in [18, Corollary 2.35]. That is we observe that after canonical identifications, ∂^π corresponds to the boundary map $\partial : H^0(K, F_*) \rightarrow H_v^1(\mathcal{O}, M_*)$ in the long exact sequence for cohomology with support. Our claim then follows from the observation that for any sheaf of rings K on a space X and K -module F , the boundary map in cohomology with support satisfies our claim. To see this, just note that multiplication by $m \in F(X)$ induces a homomorphism of sheaves $K \rightarrow F \in \text{Shv}(X)$ and consider the induced homomorphism of long exact sequences for cohomology with support.

We also have $\partial^\pi([u\pi]) = \partial^\pi(\langle u \rangle[\pi] + [u]) = \langle s^*(u) \rangle$, using [18, Lemma 3.5(1) and Proposition 3.17(3)] and the fact that the homomorphism ∂^π has kernel $\underline{K}_*^{MW}(\mathcal{O})$.

This concludes the proof. \square

Lemma 40. *If $G_* \hookrightarrow F_*$ is an inclusion of homotopy modules, then for any connected $X \in \text{Sm}(k)$ we have $G_*(X) = F_*(X) \cap G_*(k(X))$.*

Proof. Since G_* is unramified we have $G_*(X) = \bigcap_{x \in X^{(1)}} G_*(X_x)$. It thus suffices to prove the lemma in case that $X = \text{Spec}(\mathcal{O})$ with $\mathcal{O} \subset K$ a dvr with uniformizer π . Canonicity of the boundary map ∂^π implies that $\partial_G^\pi = \partial_F^\pi|_G$ and hence $G_*(\mathcal{O}) = \ker(\partial_G^\pi) = \ker(\partial_F^\pi) \cap G_*(K)$. This concludes the proof. \square

B Recollections on Continuity

We use the following results repeatedly. They are certainly very well known.

By an essentially smooth S -scheme X we mean a cofiltered diagram of S -schemes $X_\alpha, \alpha \in A$ with each $X_\alpha \rightarrow S$ smooth, and each transition map $X_\alpha \rightarrow X_\beta$ affine. Then $X := \lim_\alpha X_\alpha$. We also call $X \rightarrow S$ an essentially smooth morphism. By an essentially finite type S -scheme X we mean the same thing, except that $X_\alpha \rightarrow S$ is required to be finite type instead of smooth.

Note that it follows from [8, Théorème 8.8.2(2) and Théorème 8.10.5(v)] [24, Tags 0C0C and 01OY] that essentially smooth (respectively essentially finite type) morphisms between Noetherian schemes are stable under composition.

Lemma 41. *Let $X \rightarrow S$ be an essentially finite type morphism of Noetherian schemes. Then*

$$GW(X) \cong \text{colim}_\alpha GW(X_\alpha)$$

via the pullbacks $GW(X_\alpha) \rightarrow GW(X)$.

Proof. By [24, Tag 01ZR] the category of coherent sheaves on X is the colimit of the categories of coherent sheaves on the X_α . Any open subscheme of X is the base change of an open subscheme of X_α for α sufficiently large [8, Théorème 8.8.2(2) and Théorème 8.10.5(iii)] and hence it follows easily that the category of vector bundles (locally free finite rank sheaves) on X is also the colimit of the categories of vector bundles on the X_α . The same result for the categories of bilinear bundles is now formal, and then $K(\text{Bil}(X)) = \text{colim}_i K(\text{Bil}(X_\alpha))$.

By definition we have $GW(X) = K(\text{Bil}(X))/J(X)$, where $J(X)$ is the ideal consisting of elements $V - W$, where V, W range over metabolic bilinear bundles with isomorphic Lagrangians L [10, Section I.4]. Recall that $L \subset V$ being a Lagrangian means that $V = L \oplus L^\perp$.

It remains to show that $J(X) = \text{colim}_i J(X_\alpha)$. This is immediate from the description of the category $\text{Bil}(X)$ as the colimit of the categories $\text{Bil}(X_\alpha)$. \square

Lemma 42. *Let $s : X \rightarrow S$ be an essentially smooth morphism between Noetherian schemes of finite dimension, and $F \in \text{Pre}(\text{Sm}(S))$ a presheaf. For each α let $s_\alpha : X_\alpha \rightarrow S$ be the structure map. Then*

$$(s^* F)(X) = \text{colim}_\alpha F(X_\alpha).$$

Moreover, s^ preserves Nisnevich sheaves.*

Proof. This is a very special case of [9, Lemmas A.3 and A.4]. □

Corollary 43. *If K is a field, then $\text{GW}(K) = \text{colim}_k \text{GW}(k)$, where k runs through the subfields of K which are finitely generated over the prime subfield. (Such k in particular have finite virtual 2-étale cohomological dimension.)*

More generally, let $p : X \rightarrow S$ be an essentially smooth morphism between Noetherian schemes of finite dimension. Then there is a canonical isomorphism $p^ \underline{\text{GW}} \cong \underline{\text{GW}}$ in $\text{Shv}(\text{Sm}(X)_{\text{Nis}})$.*

The same is true for I^n or W in place of GW .

Proof. Since $K = \bigcup_k k$ we have $\text{Spec}(K) = \lim_k \text{Spec}(k)$. Also the system is filtering with affine transition morphisms. Hence the first claim follows from Lemma 41. For the parenthetical claim see [23, Theorem 28 of Chapter 4].

For the more general statement, we note that the statement with the presheaf GW in place of the sheaf $\underline{\text{GW}}$ follows from Lemmas 41 and 42. So we need to show that p^* commutes with taking the associated sheaf. This follows from [24, Tag 00WY].

Filtered colimits of abelian groups are exact, so the case of $I^1 = \ker(\text{GW} \rightarrow \mathbb{Z})$ and $W = \text{coker}(\mathbb{Z} \rightarrow \text{GW})$ follow from GW . For any filtering system (R_α, I_α) of rings with a specified ideal we get $\text{colim}_\alpha I_\alpha^n \cong (\text{colim}_\alpha I_\alpha)^n$, and hence we have established the claim about I^n . Finally the claims about $\underline{I}^n, \underline{W}$ are deduced from the results for I^n, W as before. □

Remark 44. Suitably formulated, the results in this section hold in much greater generality. We do not need this in the present article, so avoid the extra complications.

References

- [1] Jón Kr Arason. Cohomologische invarianten quadratischer Formen. *Journal of Algebra*, 36(3):448–491, 1975.
- [2] Jón Kr. Arason and Richard Elman. Powers of the Fundamental Ideal in the Witt Ring. *Journal of Algebra*, 239(1):150 – 160, 2001.
- [3] Tom Bachmann. Motivic and Real Etale Stable Homotopy Theory. *submitted*, 2016. [arXiv:1608.08855](https://arxiv.org/abs/1608.08855).
- [4] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21. Springer Science & Business Media, 2012.
- [5] Richard Elman and Christopher Lum. On the cohomological 2-dimension of fields. *Communications in Algebra*, 27(2):615–620, 1999.
- [6] Daniel Ferrand. Un foncteur norme. *Bulletin de la Société Mathématique de France*, 126(1):1–49, 1998.
- [7] Skip Garibaldi, Alexander Merkurjev, and Jean Pierre Serre. *Cohomological invariants in Galois cohomology*. Number 28. American Mathematical Soc., 2003.
- [8] A Grothendieck and J Dieudonné. Éléments de géométrie algébrique IV. *Publ. math. IHES*, 1966.
- [9] Marc Hoyois. From Algebraic Cobordism to Motivic Cohomology. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2013.
- [10] Manfred Knebusch. Symmetric bilinear forms over algebraic varieties. In G. Orzech, editor, *Conference on quadratic forms*, volume 46 of *Queen’s papers in pure and applied mathematics*, pages 103–283. Queens University, Kingston, Ontario, 1977.

- [11] Simon Krsnik. Der Multiplikative Transfer auf dem Grothendieck-Witt Ring. Diplomarbeit, Universität Bielefeld, 2006.
- [12] Kristen Luise Mazur. *On the Structure of Mackey Functors and Tambara Functors*. PhD thesis, University of Virginia, 2013.
- [13] John Willard Milnor and Dale Husemoller. *Symmetric bilinear forms*, volume 60. Springer, 1973.
- [14] F. Morel. Voevodsky’s proof of Milnor’s conjecture. *Bull. Amer. Math. Soc.*, 1998.
- [15] Fabien Morel. An introduction to \mathbb{A}^1 -homotopy theory. *ICTP Trieste Lecture Note Ser. 15*, pages 357–441, 2003.
- [16] Fabien Morel. Sur les puissances de l’idéal fondamental de l’anneau de Witt. *Commentarii Mathematici Helvetici*, 79(4):689–703, 2004.
- [17] Fabien Morel. The stable \mathbb{A}^1 -connectivity theorems. *K-theory*, 35(1):1–68, 2005.
- [18] Fabien Morel. *\mathbb{A}^1 -Algebraic Topology over a Field*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2012.
- [19] Manuel Ojanguren and Ivan Panin. A purity theorem for the witt group. *Annales Scientifiques de l’École Normale Supérieure*, 32(1):71 – 86, 1999.
- [20] Markus Rost. The Multiplicative Transfer for the Grothendieck-Witt Ring. *preprint*, 2003.
- [21] Joseph Rotman. *An introduction to the theory of groups*, volume 148. Springer Science & Business Media, 2012.
- [22] Winfried Scharlau. *Quadratic and Hermitian forms*, volume 270. Springer Science & Business Media, 1985.
- [23] S.S. Shatz. *Profinite Groups, Arithmetic, and Geometry*. Annals of mathematics studies. Princeton University Press, 1972.
- [24] The Stacks Project Authors. *Stacks Project*. <http://stacks.math.columbia.edu>, 2017.
- [25] D. Tambara. On multiplicative transfer. *Communications in Algebra*, 21(4):1393–1420, 1993.
- [26] Matthias Wendt. Units in Grothendieck-Witt rings and \mathbb{A}^1 -spherical fibrations. *arXiv preprint arXiv:1304.5922*, 2013.
- [27] Tobias Wittkop. Die Multiplikative Quadratische Norm für den Grothendieck-Witt-Ring. Diplomarbeit, Universität Bielefeld, 2006.